Xylatic Recursive Cyclo-Toral Holonic Correlatiral Holographic Trans-Synthesis

Holy Enumeration of the E_8, Leech, and Craig Manifolds, with a Working Tera-Wilson Gauge-Holonic Bennett-Reversible Turing-Machine Architecture as an Agnostic Omni-Holographic Linguistical Substrate

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Abstract

We construct, enumerate, and operate a finite symbolic substrate in which thermodynamic, quantum-boundary, topological, computational, and linguistic structure are recovered as exact ledger readings of one another. The whole framework compresses to *one equation in one free choice*: $S^2 = p\,\varphi^6/(2h)$, with the twin-prime seed $(3, 5)$ the only parameter — five constants collapse to the seed, the kissing number is seed arithmetic ($K_0 = 2(p^2-1)pq = 240$), and the Möbius recursion $p_{n+1}^2 = p_n^3 - p_n + 1$ generates and terminates (four independent proofs) a three-storey tower of exceptional objects — the E_8 root system (dimension 8), the Leech lattice Λ_{24} (dimension 24), and a Craig-type lattice (dimension 120) — together with their carrier polytopes, clocks, symmetry orders, and censuses. The paper's architecture is a declared list of *twelve dynamics* — engine, projection, fold, cancellation algebra, quantum minimum, enumerations, σ-flow, accounting, instrument, spine, language, observer — each owning a section, every claim serving one of them (§1.6). Part I is the construction: the *cross-parity projection* — the E_8 root system read in the Coxeter eigenframe of its chirality twin — derives a six-shell grammar $r_k^2 = 1 + k\sqrt{5}/10$ with palindromic populations $[24, 56, 40, 40, 56, 24]$ and perfect integer/spinor type separation, with the mechanism exhaustively characterized (all 128 odd mirrors separate, all 128 even mirrors fail; a Galois selection rule picks the plane pair; the construction is E_8-special); the same mirror fixes the 84-root anchor manifold the thermodynamic ledger later prices. The *dualistic minimum* supplies the quantum closure: the diagonal lift closes exactly at the Tsirelson boundary, $S = 2\sqrt{2}$ in $\mathbb{Q}[\sqrt{2}]$, with the identity $S_q^2 = S_c \cdot S_a$. Part II delivers the enumerations (the full Leech first shell from a verified Golay code; the categories $720 + 90{,}720 + 105{,}120$; the block-activity law $45 \cdot 16^k$; the exact Craig trace form) and the exact information thermodynamics: the 84-bit ledger, the prime-logarithm entropy bookkeeping, and a concentration theorem localizing all irreversibility on the two $|k| = 2$ shells in the exact ratio $9:5$. Part III presents the working architecture: the hyperbinary operator algebra $\mathrm{Lips}_8 \cong (\mathbb{Z}/2)^3$ through which every state mutation flows; the Tera-Wilson dual-rail system, in which an abelian gauge holonomy certifies paths (mirror-blindness is a theorem) while a braid lift preserves the handedness the gauge provably cancels; and a Bennett-reversible machine whose single irreversible instruction carries the 84-bit price. Throughput is measured at

[1] On terminology: *holy* (for the Golay-frame decompositions of the Leech shell, after the classical "holy constructions" of Conway–Sloane), *Hagiography* (the program's internal name for its structural inventory volume), *correlatiral* (a coinage: correlation- carried denotation), and *Tera-Wilson* (recording the 10^{12} ops/s regime in which gauge certification was first demonstrated) are program-internal names retained for continuity with the source corpus; they carry no doctrinal content.

7.6×10^{11} bit-deterministic gauge operations per second on a four-core commodity host in this work (2.9×10^{12} on 32 cores in the pinned record), with canonical digests byte-identical across hardware and across a two-month interval. Part IV states the paper's thesis — **language is holographic symbolic exchange, and holographic symbolic exchange is the base of language** — at exactly its audited strength: the anchor partition is coordinate geometry, the semantic router carries it one-hot by construction, and the independence audits that separate architecture from measurement are reported in full, including a refuted conjugate-frame hypothesis and a no-silent-damage certificate. Part V closes — deliberately last — with the physics reading (boundary incompleteness), the mind conjecture stated as conjecture with a falsifiable thermodynamic edge, and the honesty boundary. A selection-and-termination synthesis completes the mathematics: the unit-form spectrum law is proved for every squarefree odd conductor (the master theorem); the design–genus correspondence closes at a complete, falsifiable boundary (twelve genus-zero Atkin–Lehner conductors among 84, every one a Monster class order); and the moonshine family ladder with its top-prime law — exact at eight sporadic groups, adjudicated against a pre-registered deflation null — joins the two object-side selection theorems. Every claim in the paper carries a tier, a magnitude calibration, and a named validating artifact from the audited laboratory corpus.

Contents

1 Introduction

1.1 What this paper claims, and for whom

This paper is written for the skeptical reader. It unifies a research program whose parts have circulated as separate working notes, and its claims span mathematics, information theory, computer systems, and — at its outer edge, explicitly labeled — philosophy of mind. Such a span invites two opposite failure modes: the enthusiast's, in which analogies are silently promoted to identities; and the skeptic's, in which exact results are dismissed because they arrive in unfamiliar company. Our defense against both is the same instrument: a strict, machine-checked claim stratification (§1.4) under which every numerical statement in this paper carries an explicit tier, every **[EXACT]** statement is re-derivable by exact arithmetic from the definitions given here, and every **[MEASURED]** statement names its protocol, hardware, and date. Nothing in a lower tier is ever used as a premise for a higher one.

The claims, in ascending order of ambition:

1. **Mathematics [EXACT]**. From the twin-prime seed $(3, 5)$ a three-storey tower of exceptional structures is forced — E_8, Leech, Craig — with all dimensions, clocks, censuses, and symmetry data derived, not chosen (§6). The Leech first shell is enumerated vector-by-vector from a Golay code constructed and verified from its generator polynomial (§7); new exact results include the block-activity law $45 \cdot 16^k$, the prime-logarithm entropy closure, the $9 : 5$ concentration theorem, and the Tsirelson factorization $S_q^2 = S_c\, S_a$. A selection-and-termination synthesis (§17) then proves the unit-form spectrum law for every squarefree odd conductor (the master theorem, C69), computes the complete design–genus boundary over 84 conductors (twelve genus-zero, all Monster class orders — C71), and adds a group-side pillar: the moonshine family ladder and the top-prime law at eight sporadic groups, adjudicated against a pre-registered deflation null (C72–C76).

2. **Physics reading [READING] with exact skeleton [EXACT]**. Boundary incompleteness — built on Poynting, Landauer, Noether, and the holographic bounds — explains *why* a finite observer's physics is a boundary ledger; the substrate supplies the exact discrete grammar such a ledger obeys (§15).

3. **Computer systems [MEASURED]**. A working architecture — the hyperbinary mutation membrane, dual-rail addressing and holonomy, reversible computation with priced commits — measured at record-grade, bit-deterministic throughput with cross-hardware, cross-time byte-identity (§11–§12).

4. **Linguistics [EXACT]** census, **[READING]** thesis. The substrate's semantic bus is carried *one hundred percent* by its crystallographic gauge — a measured, zero-mixing stratification — grounding the thesis that language is holographic symbolic exchange (§13).

5. **Mind [CONJECTURE]**. A conjecture, clearly labeled, with a stated falsification protocol (§16).

Three reading paths. The scope is wide by design; the paper is organized so that three readers can each take a closed path. The *mathematician*: §3–§4, §7–§9, then §17 (the construction, enumerations, censuses, exact identities, and the selection/termination synthesis stand alone). The *systems reader*: §6.3, §10–§12 (the architecture and the measured record). The *foundations reader*: §15, §13–§16. One invariant holds on every path: *the exact spine survives if every sentence* *tagged **[READING]** is deleted* — readings interpret the mathematics; they are never premises for it.

1.2 The claims, their magnitude, and their validation

Beyond tiering, every claim carries a *magnitude* calibration — what kind of novelty is being asserted — and names its validating artifact in the accompanying suite (gate counts in parentheses; the appendix volume maps artifact names to runnable code and data): *new mathematics* = a statement we have not located in the literature (stated as such, never as a priority claim); *new here* = new to this program, plausibly known elsewhere; *classical* = known mathematics recomputed under the exact discipline; *architecture* = a fact of the built system, true by construction and stated as design, never as discovery; *measured* = an engineering measurement with protocol, hardware, and date; *reading/conjecture* = interpretive, quarantined. The table is machine-generated from the provenance trace (C01–C76) and the body sections cite these identifiers.

id	claim	tier	magnitude	validation (gates)
C01	six-shell law $r_k^2 = 1 + k\sqrt{5}/10$, populations $[24, 56, 40, 40, 56, 24]$	**[EXACT]**	new here	`cross_parity_projection` (7/7)
C02	D_8/S^+ type separation read off shell parity	**[EXACT]**	new here	`cross_parity_projection` (7/7)
C03	parity switch: all 128 odd mirrors separate; all 128 even fail	**[EXACT]**	new math	`cross_parity_projection` (7/7)
C04	Galois selection rule: exactly 2 of 6 pairings separate	**[EXACT]**	new math	`cross_parity_projection` (7/7)
C05	E_8-speciality: $2 = d/4$ only at $d = 8$	**[EXACT]**	new here	`cross_parity_projection` (7/7)
C06	τ_8 fixes the 84 anchors / permutes $D28$ / everts $S128$	**[EXACT]**	new here	`shell_involution_register_audit` (8/8)
C07	concentration 54:30 = 9:5 on $\lvert k\rvert = 2$; $A24$ 3:1; $F60$ 3:2	**[EXACT]**	new math	`shell_involution_register_audit` (8/8)
C08	84-bit ledger $480 - 396 = 84$; Bennett split	**[EXACT]**	new here	`information_thermodynamics_paradigm` (11/11)
C09	exact prime-log entropy closure, zero residual	**[EXACT]**	new here	`claim_provenance_trace` (5/5, geometric re-derivation)
C10	prime-flow atlas: state vs. path; foreign primes net zero	**[EXACT]**	new math	`entropy_prime_flow_atlas` (6/6)
C11	tower identities $\varphi/\sigma/\mathrm{ord}$ over 15/35/143	**[EXACT]**	new here	`order_algebra_registers` (39)
C12	eversion $M^h = -I$ on all $\varphi(h)/2$ planes	**[EXACT]**	new here	`eversion_facet_dynamics` (6/6)
C13	24-cell $GF(2)$ ranks $(23, 73, 23)$, Betti $(1, 0, 0, 1)$	**[EXACT]**	classical	`sigma_betti_conservation` (7/7)
C14	$\sigma \leftrightarrow \beta$ bridge $76+176 = 252 = \sigma(96)$; $(v-1)(v-4)/2$	**[EXACT]**	new math	`sigma_betti_conservation` (7/7)
C15	breath calculus $\Omega = \varphi \circ \sigma$: 8 anchors, 4 laws	**[EXACT]**	new math	`capstone_laws_enumeration` (4/4)
C16	Leech shapes $97152 + 1104 + 98304$ from verified Golay	**[EXACT]**	classical	`master_polytope_lattice_enumeration` (19/19)
C17	holy block-activity law $45 \cdot 16^k$	**[EXACT]**	new math	`leech_per_vector_holy_membership` (9/9)
C18	standard frame fails holy $A/B/C$ (negative, retained)	neg.	new here	`de_forms` (18/18 + mismatch)
C19	Craig trace form; σ_{12} fixed-point-free; K_2	**[EXACT]**	new here	`craig_120d` + `craig_l2_120` (6/6, 12/12)
C20	CHSH closes at $2\sqrt{2}$ in $\mathbb{Q}[\sqrt{2}]$; $S_q^2 = S_c S_a$	**[EXACT]**	classical	`quantum_symbolic_closure` (6/6)

id	claim	tier	magnitude	validation (gates)
C21	gauge mirror-blind; Burau mirror-faithful	**[EXACT]**	new here	`reversible_ledger_machine` (21/21)
C22	closure chirality censuses (120/120; class skews)	**[EXACT]**	new here	`claim_provenance_trace` (5/5, geometric re-derivation)
C23	router one-hot $V_4 \to$ anchor	arch.	architecture	`stratification_independence_audit` (4/4)
C24	anchor partition is coordinate geometry (240/240)	**[EXACT]**	new here	`stratification_independence_audit` (4/4)
C25	observer rail = constructed frame; same-index identification refuted; no silent damage	neg.	new here	`shell_involution_register_audit` (8/8)
C26	byte-identical replay across vendors and months	**[MEASURED]**	measured	`walker_sentence_weld` (9/9)
C27	throughput record + fresh re-run, determinism-first	**[MEASURED]**	measured	pinned claims + 2026-07-08 re-run
C28	linguistic thesis; mind conjecture; boundary reading	**[READING]**/**[CONJECTURE]**	reading	stated in §13–§16
C29	the One Equation $S^2 = p\varphi^6/(2h)$; five constants collapse to the seed	**[EXACT]**	new math	`information_thermodynamic_ledger` (13/13) + `tower_role_relativity` (7 windows)
C30	singularity cubic $u^3+3u^2-13u+1=0$ derived from $S(q)=1$; root in $(\sqrt{5},\sqrt{7})$; metallic family: only the golden member is prime	**[EXACT]**	new math	`metallic_delta_family` (8/8)
C31	moments are dimensions: $M_2/M_0 = 4$, $M_4/M_0 = 24$, variance $= 8$; structured failure M_6/M_8 excesses $8 \cdot 7$, $8 \cdot 143$; $Z(\ln\varphi) = 64\varphi$	**[EXACT]**	new math	`moment_dimension` (9/9)
C32	exponent sieve 29→15→10→8; prime resurgence into the populations; cyclotomic defect zero (scope disclosed)	**[EXACT]**	new here	`sieve_cascade_resurgence` (7/7)
C33	CPT triple priced at L_0: σ_6 transparent (0 bits), σ_{34} shares the 84-frame; kernel $\{1, \sigma_6\}$	**[EXACT]**	new math	`cpt_landauer_costs` (6/6)
C34	golden fold charted: type = icosian lock; five golden norm classes $[24, 24, 144, 24, 24]$; rational nucleus = inner 24-cell; join obstruction 0/8	**[EXACT]**	new math	`golden_join_materialization` (7/7)
C35	Leech $\sqrt{5}$-type census 37,800/120,960/37,800; V_4-invariant; charged:neutral = 5:8 in 13ths	**[EXACT]**	new math	`leech_pervector_type_census` (17 checks)
C36	three coexisting per-vector partitions of the 196,560, agreeing exactly on the 720 core	**[EXACT]**	new here	`leech_per_vector_holy_membership` (9/9) + C17/C35 labs
C37	Craig 120-unit layer complete: phase table, coordinate dump, icosian realization; shells $6{\times}20$ uniform (not palindromic); $K_2 =$ 160,982,640 tower-exact	**[EXACT]**	new here	`craig_120d` (28/28) + coordinate dump (6/6)
C38	neighbor censuses: Leech 1+4600+47,104+93,150+ . . . ; Craig Ramanujan spectrum $\{120, 1, -10, -12\}$, counts 1+100+10+9	**[EXACT]**	classical/new here	`de_forms` (18/18) + `craig_l2_120` (12/12)
C39	anchor migration $(84, 78) \to (0, 732) \to (0, 60)$; compactification base-exclusive; apex standing wave $\Sigma t_5 = 0$	**[EXACT]**	new math	`h_motion_leech_apex` (7/7)

id	claim	tier	magnitude	validation (gates)
C40	CPT profile theorem: T fixes all 1464 profiles, C/P fix none; transparent generator switches P→T; $819{=}3(14^2{+}7^2{+}28)$; $27{=}3^3$ grammar	**[EXACT]**	new math	`leech_L1_galois_grammar` (8/8)
C41	Niemeier ladder $\theta_N{-}\theta_{\mathrm{Leech}}{=}24h_N\Delta$; zero-point $720{=}24{\times}h_C(E_8)$; $4600{=}K(\Lambda_{23})$	**[EXACT]**	new here	`niemeier_theta_ladder` (6/6)
C42	boundary kernel: equilibrium →84-floor, reversible null =0, tower profile $\{84, 0, 0\}$; apex address space 732 pairs, V_4-invariant 27-grammar syndrome, zero-heat capacity	**[EXACT]**	new math	`boundary_response_kernel` (7/7) + `apex_pair_addressing` (6/6)
C43	the modular index ladder: $\psi(h){=}\sigma(h){=}[\mathrm{PSL}_2(\mathbb{Z}){:}\Gamma_0(h)]$; Fricke genus zero exactly at $h{=}15, 35$; Ogg fold at 1321	**[EXACT]**	new math	`moonshine_genus_ladder` (6/6)
C44	ABSOLUTE termination: full Atkin–Lehner genus $\{0, 0, 1, 5, 12, 49\}$; $X_0(143){+}$ elliptic; unrealized-storey genera $1, 5, 12$ = pentagonal $P(1..3)$, fold breaks at 7^2	**[EXACT]**	new math	`al_quotient_ladder` (6/6)
C45	two hauptmoduls per storey: sum rail $x{+}1/x$ (Fricke/B class), difference rail $1/x{-}x$ (full-AL/A class); prime mirror = fold sign -1 (exact Dedekind sums); $T_B^2{-}T_A^2{=}4{=}\lvert V_4\rvert$ identically; T_{15A} opens 8, 30, 72	**[EXACT]**	new math	`hauptmodul_weld` (7/7)
C46	the angle system: plane budget 384=360+24 split 144+240; Leech cosines exactly $\{0, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm 1\}$ (the quarter as the generic neighbor); Craig cosines are reciprocal constants	**[EXACT]**	new math	`angle_atlas_chain` (8/8)
C47	Stokes = Legendre pair $(k\vert p, k\vert q)$: the plane distributions derived from character theory; class sums carry 23, $1539{=}81{\cdot}19$, $20{=}1/\delta^2$	**[EXACT]**	new math	`stokes_design_angle_theorems` (8/8)
C48	design-moment forcing and breaks: censuses are the unique moment solutions; breaks at the invariant degrees with excesses $(3/4)^2$ and $(3^4 5^2{\cdot}23)/(2^{12}{\cdot}17)$; Craig unit proxy not a 2-design (599/3600)	**[EXACT]**	new math	`design_moment_break` (6/6)
C49	the cusp-form mechanism (Venkov, machine-verified) + bridge inventory ($K_1{+}324{=}j_1$; $744{=}\sigma(240)$; $196883{=}K_1{+}323$; Conway divisibility by 324) + the level-1-only match-point explained by C44	**[EXACT]**	new welds	`design_genus_termination_weld` (8/8)
C50	selection, both families: Leech unique among the 24 Niemeier lattices with mechanism-forced 11-design shell ($a_1{=}24h_N{\neq}0$ else); genus-zero exactly $\{15, 35\}$ among ALL fifteen twin-prime conductors to 39,203	**[EXACT]**	new math	`tower_extremal_selection` (4/4)
C51	out-of-sample survival: the $h{=}323$ and $h{=}899$ unit forms constructed (general spectrum law $\{1, \frac{1}{\varphi(h)}, -\frac{1}{\varphi(q)}, -\frac{1}{\varphi(p)}\}$, counts twin-shifted $(p{-}2)(q{-}2), q{-}2, p{-}2)$; both fail 2-design; both points on-diagonal	**[EXACT]**	new math	`second_crystal_design_object` (6/6)

id	claim	tier	magnitude	validation (gates)
C52	T_2-replication separates the rails: A-rails replicate coefficientwise to q^{14} (zero defects); B-rails fail with first defect -2 at q^2 both storeys; the weight-2 eta form at 143 has Fricke sign -1; $\dim S_2 = 13 = 11+1+1$ locates the elliptic differential	**[EXACT]**	new math	`mckay_thompson_series_map` (5/5)
C53	the deposition engine: one σ-step kernel reproduces both canonical chains and the frozen σ-graph edge-for-edge (63/63); bridge/orphan detectors fire at $\sigma(60)=\sigma(143)=168$ and 323; self-halts at the 1321 fold; mediate lands on the self-dual carrier (palindrome $(23, 73, 23)$ from V, E alone)	**[EXACT]**	new math	`conceptualization_deposition_kernel` (8/8) + `kernel_standing_wave_weld` (4/4)
C54	the amortization dichotomy: all five named re-encodings are computed bijections with zero-bit erasure census; one priced act (the eversion); amortized extraction cost $\to 0$; universal closure open	**[EXACT]**	new math	`htce_amortization_theorem` (7/7)
C55	the certified-substrate benchmark: 100% detect + 100% address recovery over all 49,776 corruptions vs baseline 96% silent misroute (= $1-1/27$ collision arithmetic, $+0.4\sigma$; gap dynamics excluded at -15σ)	**[MEASURED]**	applied record	`semantic_advantage_benchmark` (6/6) + `convexity_gap_identity` (5/5)
C56	the convexity-gap identity: $\varphi^3+\varphi^{-3}=2\sqrt{5}=1/\delta$, so $S=\delta\varphi^3$ and $1-S=\delta\varphi^{-3}=(5-2\sqrt{5})/10$ (the 5.3% gap, exactly); conserved product $\delta^2=1/20$ the corollary; 4/75 rejected	**[EXACT]**	new math	`convexity_gap_identity` (5/5)
C57	the frame-square bracket: $h = f^2-1$ at all five storeys, so McKay's $+1$ is the frame identity read at $(17, 19)$; $j_1 = 12^2{\cdot}1365+18^2$ over consecutive frames (a boundary phenomenon of the termination, not a series — see C64)	**[EXACT]**	new math	`frame_square_mckay_bracket` (4/4)
C58	the excess cubic: unit-form 2-design excess $= N(p)/d^2$, $N(p)=2p^3-p^2-4p-2$, $d=p^2-1$; all three reduced numerators prime; the 499/599 near-twin dissolved (11/1600 exactly)	**[EXACT]**	new math	`excess_numerator_denotation` (5/5)
C59	the pairing inequivalence: one Ramanujan kernel, two index geometries; the unit form vertex-transitive, the trace Gram not; row-spectra multiset a relabeling invariant $\Rightarrow$ no bijection; the moved pair located at exponents $\{121, 132\}$	**[EXACT]**	new math	`craig_pairing_reconciliation` (5/5)
C60	the over-determination theorem: all derivation mechanisms for K_0, K_1 agree at the realized storeys and terminate simultaneously at L_2; $2d(2^{d/2}-1)$ at $d=120$ contradicts the 819 cascade (crossing exactly at $d=24$); no Craig kissing number asserted	**[EXACT]**	new math	`craig_kissing_discriminant` (5/5)

id	claim	tier	magnitude	validation (gates)
C61	the nesting asymmetry: every enumerated constructor small→large (θ_{Leech} a polynomial in θ_{E_8}; lift bijective); downward maps forget ($\{0, 16, 96\}$=112; anchors $(84, 78)\to(0, 732)\to(0, 60)$); the imprint classified; constructive vs epistemic axes sort with zero counterexamples	**[EXACT]**	new math	`nesting_asymmetry` (5/5)
C62	the anti-pairing mechanism: both runtime mirrors fixed-point-free involutions with balanced displacement histograms (the void slot = fixed-point-freeness); zero $PGL(2, 7)$ realizations (exhaustive); 28 = $D28$ coset involutions; the fixed 24-cell center derived	**[EXACT]**	new math	`mirror_antipairing_identification` (5/5)
C63	rungs 4–5 of the unit-spectrum law by full enumeration (h=1763, 3599; both on-diagonal — seven points, zero violations) and the streak answer: the excess-cubic numerators go composite at (41, 43) and (59, 61) — C58's three-storey primality was coincidence, honestly closed	**[EXACT]**	new math	`second_crystal_next_rungs` (4/4)
C64	the level-2 termination is two-sided: j_2 admits no tower decomposition (K-side offset 139,488,880 structureless vs. level 1's 324) while McKay's level-2 identity $j_2 = d_2 + d_1 + 1$ is Monster-side exact — the C57 economy fails at j_2, as the match-point mechanism predicted	**[EXACT]**	new math	`moonshine_level2_divergence` (4/4) + `monster_side_level2` (4/4)
C65	the drag reconstruction: from the committed void table and one stated doubling assumption, the counter-orbit array $\{\pm 2, \dots, \pm 24\}$ (one each, void slot empty) is derived in-tree; all 50 symmetry-respecting isometries preserve it, one broken pairing destroys it (identification with the lost source document stays reconstruction-dependent)	**[EXACT]**	new here	`constellation_drag_reconstruction` (4/4)
C66	the σ-chain theorem: from the L_1 seed $C(24, 2)$=276, σ^3=6552=$K_1/30$ and σ^4=21,840=$K_1/9$ — both writings of the kissing number, with the E_8 design moments as cofactors; seed-stamp census: every storey seed above the base carries a named upward prime ($276 = 12 \cdot 23$)	**[EXACT]**	new math	`sigma_chain_L1` (4/4) + `sigma_exit_census` (3/3)
C67	the general-pq spectrum theorem: the C51 law proved for all distinct odd primes $p < q$ (three elementary lemmas; verified over 156 conductors) — the out-of-sample instrument no longer rests on enumeration	**[EXACT]**	new math	`general_pq_spectrum_proof` (4/4)
C68	the three-prime spectrum theorem: at h=pqr the 8-value spectrum with Möbius sign lattice $\mu(h/g)$ and $(s-2)$-product counts; excess at 105 is $37{\cdot}43/48^2$ (does not price out — honest miss; twin-rung pricing is special)	**[EXACT]**	new math	`pqr_spectrum_theorem` (3/3)

id	claim	tier	magnitude	validation (gates)
C69	the master theorem: for every squarefree odd h the unit-form spectrum is the 2^k divisor classes $g \mid h$ with value $\mu(h/g)\,\varphi(h)/\varphi(h/g)$ and count $\prod(s-2)$; class census the invariant (value-merge first at 1365: 15 values, 16 classes); k=4 verified by full enumeration	**[EXACT]**	new math	`squarefree_master_theorem` (4/4)
C70	the scoped correspondence: h=105 is the first off-diagonal point (not design-maximal, genus zero) — the naive biconditional is false in general; among the eleven three-prime conductors $\leq$ 400, off-diagonal $\Leftrightarrow$ Monster class order (11/11 on that sample; the full-range law is the one-directional C71); the k=4 forecast survives ($g^+ = 8, 9, 12, 14$; zero new off-diagonal points)	**[EXACT]**	new math	`three_prime_correspondence` (3/3) + `squarefree_master_theorem` (4/4)
C71	the complete boundary: over all 84 composite squarefree odd conductors $\leq$ 400, exactly twelve genus-zero full-AL quotients, all ≤ 119 and all Monster class orders (12/12; one-directional — the converse fails at $57, 93$); design-maximal $\Leftrightarrow$ realized storey in-category; the tower is the intersection	**[EXACT]**	new math	`converse_hunt` (3/3)
C72	the deflation null instrument: condition on 15-SS-smoothness in $[d/2, 2d]$, enumerate exactly, pre-registered decision bands — the Monster ladder property is selective at 2.99×10^{-7}, the 13^2 pairing weakly selective, per-irrep singles and fiber pricing undecided (honest verdicts both ways)	**[EXACT]**	new math	`ss_smooth_deflation` (3/3)
C73	the supersingular head-character ladder: all six Monster head dims factor over the 15 supersingular primes, each carrying $\geq$ 2 unrealized-rung twin primes (outer twins never appear); family-wide with group-specific signatures — B carries $\{23, 31, 47\}$, selective at 3.98×10^{-7}; T_{2B}'s first coefficient is 276	**[EXACT]**	new math	`monster_irrep_ladder` (4/4) + `mckay_thompson_rung_content` (4/4)
C74	the top-prime law at eight groups: d_1 = (framework cofactor) $\times$ (product of $\lvert G\rvert$'s largest primes) — M: $47{\cdot}59{\cdot}71$ (1); B: $3{\cdot}31{\cdot}47$; Fi24′: $13{\cdot}23{\cdot}29$; Fi23: $2{\cdot}17{\cdot}23$; Co1: $12{\cdot}23$; Co2/Co3/M24: 23; cofactors $\{1, 2, 3, 12, 13\}$ all framework constants (orders and minimal dimensions cited from the ATLAS [19]; factorizations exact); the 276 triangle: σ-seed = T_{2B} coefficient = $d_1(\mathrm{Co1}) = C(24, 2)$	**[EXACT]**	new math	`signature_law` (3/3) + `top_prime_law_extension` (3/3)

id	claim	tier	magnitude	validation (gates)
C75	the cofactor law's honest closing: the moment-cofactor identity is $E_8 \to$Leech-exclusive (census over eight families); the σ_3/E_4 route is a total miss (exhaustive preregistered grammar); one-level selection with complete bookkeeping ($819 = \sigma(2^5)\sigma(3^2)$; bridge prime $= \sigma(9)$), mechanism-less and posted	**[EXACT]**	new math	`cofactor_mechanism_census` (3/3) + `sigma3_route_verdict` (3/3) + `sigma_orbit_mechanism` (4/4)
C76	the constraint-null degeneracy: the McKay system is triangular with unit diagonal, so j + the multiplicity pattern determine the head dims uniquely (reconstructed exactly) — a "constraint-aware null" has nothing to sample; the independence-null verdicts stand as verdicts about j's own arithmetic	**[EXACT]**	new math	`constraint_null_resolution` (3/3)

Table 1: The master claims table: 76 rows, each naming its validating artifact (292 gated artifacts across 148 laboratories, 1,062 recorded gates, all passing at the 2026-07-12 governance audit). Every **[EXACT]** row is provenance-closed on the geometric rail (§14). Anchor-class symbols $A24/F60/D28/S128$ (the four V_4 strata of the 240 roots) are constructed in §7.

1.3 Relation to the three source monographs

This paper is the unification, written to be read *without* its predecessors. Three monographs stand behind it: *Linguisticus Toroidalficarum* (the machine: the working system, its grammar, its reversible ledger, its benchmarks), *Philosophiæ Naturalis Principia Linguistica* (the physics reading: holographic-language primacy, boundary-disclosed information, the designed six-experiment spine), and the *Hagiography of the Exceptional Manifolds* (the structure: the complete per-object inventory of the polychora, the Weyl groups, and the three lattice storeys, every datum tiered). Volume II adds no mathematics beyond Volume I — the two are deliberately held to different standards (the machine is proved; the reading is read) — and the Hagiography is their consolidated structural record, carried here as the appendix volume. The present paper subsumes the load-bearing content of all three, restates every prerequisite from scratch (§3), and adds the new exact results of the current cycle: the prime-logarithm entropy closure, the concentration theorem, the Tsirelson factorization, the breath calculus, holographic thermal correlation extraction, the fresh benchmark record, and the One-Equation front matter (§2), the singularity-cubic and metallic-family gates, the moment–dimension and sieve–resurgence theorems, the per-generator CPT ledger, and the golden-fold census with its join obstruction (§4.3).

1.4 The claim tiers

tag	discipline
[EXACT]	proved by exact arithmetic or exhaustive finite census; no floating point participates in any gated value; re-derivable from the definitions in this paper
[MEASURED]	an executed, recorded measurement; protocol, hardware, and date stated; deterministic replays where applicable
[PREDICTED]	forced by an exact pattern and validated on a known case before being asserted on a new one
[READING]	a disciplined interpretation *of* exact structure; never used as a premise
[CONJECTURE]	a conjecture stated in falsifiable form
[OPEN]	an open problem, named

Beyond tagging, the corpus behind this paper is governed mechanically: an automated audit of the program's evidence graph verifies that every passing verdict is backed by all-true gate records, that no result cites a nonexistent predecessor, and that no composition claims a tier stronger than its weakest member. At the time of writing that audit covers 292 gated artifacts across 148 laboratories, 1,062 recorded gates, all passing, with zero integrity violations **[MEASURED]** — including the step-one cross-parity construction (Theorem 4.3) and the independence audits of Remarks 13.2 and 9.5. Honest negatives are retained in place and appear in this paper where they matter (§7.4, Remark 13.2).

1.5 Terminology

The program's vocabulary is unusual; we define it before use.

term	meaning
Xylatic	of the crystal substrate: the exact, seed-forced lattice fabric all member calculi compute over
cyclo-toral	the address space is a discrete Clifford torus $T^2 = S^1_5 \times S^1_7$ traversed by modular clocks; recursion lifts it level to level
holonic	every carrier is simultaneously a whole and a part: a local symbol object participating in global correlations
correlatiral	meaning and state are read as correlations across addresses (cross-denotation), not stored at locations
trans-synthesis	the non-sequential construction of addresses: content is located by computing where it must be, not by searching
holographic	every fragment carries a projection of the whole; access is boundary-limited; readings are interference phenomena
hyperbinary	the layered operand fabric of §6.3: atom bits $\times$ ternary reliability $\times$ senary shell $\times$ quaternary gauge $\times$ septenary channel $\times$ sexagesimal relation $\times$ 25-fold perspective
eversion	the half-period antipode $M^h = -I$: the global inside-out involution that is the substrate's one irreversible act
holy (enumeration)	the Golay-frame decompositions of the Leech shell (categories $A/B/C$; the E_8^3 trio frames)
anchor	the $84 = 24 + 60$ involution-fixed roots of E_8 (the $A24$ core and $F60$ free anchors)
commit	the priced, non-bijective state transition; everything else is reversible

1.6 The twelve dynamics

The program is not a heap of results. Everything this paper proves, measures, or reads sits on **twelve dynamics**, and the paper's architecture is this list — each dynamic owns a section, and every row of the claims table serves one of them. Stated once, up front, at one sentence each:

1. **The one-parameter engine** (§2) — one equation, one free choice: $S^2 = p\varphi_q^6/(2h_C)$ collapses five constants to the twin-prime seed, and the Möbius recursion $p_{n+1}^2 = p_n^3 - p_n + 1$ generates the tower and terminates it four independent ways. [C29–C32]

2. **The cross-parity projection** (§4) — reading E_8 in the eigenframe of its parity-broken chirality twin derives the six palindromic shells, the free type bit, the parity-switch and Galois-selection mechanisms, and the emergence of the number field itself. [C01–C05]

3. **The golden fold** (§4.3, §6) — the icosian second lane: type = integer lock, five golden norm classes, the hyper-chirality plane spectrum $4 \to 12 \to 60$ multiplying by the seed primes, and the join obstruction that sharpens the two-rail question. [C34]

4. **The V_4 cancellation algebra** (§4, §6) — symmetry as cancellation: the Klein four-group is the Galois group $\mathrm{Gal}(\mathbb{Q}(\sqrt{5}, \sqrt{7})/\mathbb{Q})$, acts as the Stokes polarization algebra, recurs at every level, and its modular realizations *are* the carrier polytopes. [C11]

5. **The dualistic minimum** (§5) — the diagonal lift $2 \to \sqrt{2} \to 2\sqrt{2} \to 4$ closing exactly at the Tsirelson boundary, with the metallic family selecting the golden member uniquely. [C20, C30]

6. **The full enumerations** (§7) — E_8, Leech, and Craig materialized, not sketched: the 196,560 built exactly with three coexisting per-vector partitions, the theta series past the first shell, the Craig 120-unit layer complete, the kissing tower $K_n = 240 \cdot 819^n$. [C16–C19, C35–C38]
7. **The epitaxial and σ-flow dynamics** (§8) — structure deposited in strict golden order; the divisor-sum flow with its orphan sources, attractor sinks, and H-theorem; the breath calculus. [C13–C15]
8. **The Landauer accounting** (§9) — the 84-bit rigidity schema (fixed count *is* the bit budget), dimensional reduction as literal erasure, the per-generator CPT prices, and HTCE: observation is free, only the eversion pays. [C08–C10, C12, C33]
9. **The Fisher–Gram instrument** (§9) — the 2-design theorem chain making the information metric exactly $\frac{1}{4}I = I/|V_4|$, flat, tower-invariant: free motion is a consequence, not a postulate. [within C08's weld; Theorem 9.6]
10. **The commutator spine and the dual rail** (§10–§11) — the order-blind measurement and the order-sensitive memory factor through the commutator subgroup; what the gauge cancels the braid keeps; the realized registers carry the eversion as commit; the Bennett machine runs it. [C21–C22, C26–C27]
11. **The language stratification** (§13) — the semantic bus is 100% carried by the V_4 classify gauge, one-hot, measured; the morpheme stratum $81 = 3^4$ equals the V_4 orbit count on the roots; the walker weld makes the event stream a sentence, byte-identically. [C23–C24]
12. **The observer and the void** (§14, §16, §8) — the seat of change is the empty center; the rook chart, the shadow theorem (two distinct 84s meeting in 26), the two rails held apart by audit, and the law of subjectivity. [C06–C07, C25, C28]

The bracketed ranges above name each dynamic's charter claims; the later rows of the claims table distribute across the same twelve — the selection, termination, and moonshine spine (C39–C76, §17) serves the engine (1), the enumerations (6), the σ-flow (7), and the instrument (9), with each result's dynamics assignment carried in the program registry. Nothing in the paper stands outside this list; the physics reading (§15) and the mind conjecture (§16) are readings *of* these twelve, quarantined at their own tier. The paper states this center first and hangs everything on it.

1.7 Roadmap

The paper is engine-first, then construction. §2 opens with dynamic 1 — the One Equation $S^2 = p\varphi_q^6/(2h_C)$, the five-to-one parameter collapse, the Möbius recursion with its four termination proofs, the singularity cubic, and the moment/sieve results that show the engine acting. Part I (§3–§6) then builds the mathematics from nothing: preliminaries; *the cross-parity construction* (the projection, the parity switch, the type separation, the anchor-fixing mirror, and the golden fold with its five-norm census — dynamics 2–4); the dualistic minimum with its exact quantum closure and palindrome family (dynamic 5); the one-seed tower and its phylum of calculi, including the hyperbinary operator algebra. Part II (§7–§9) is the exact body — dynamics 6–9: the full enumerations; the epitaxial and σ-flow dynamics; the information thermodynamics with the Fisher–Gram instrument. Part III (§10–§12) is the machine — dynamic 10: the dual-rail architecture, the reversible Turing machine, and the measured record — architecture and measurement, scoped as such. Part IV (§13–§14) states the language result (dynamic 11) at its audited strength and then documents the audits and the rail discipline (dynamic 12) themselves. Part V (§15–§20) carries — deliberately last — the physics reading, the subjective dynamics and the mind conjecture, then the selection-and-termination synthesis (§17: the design–genus boundary, the master theorem, and the moonshine family ladder — the paper's deepest mathematics, placed here because it welds every earlier face), the candidate laws, related work, and the honesty boundary.

The appendix volume that accompanies this paper is not supplementary color; it is the program's *complete structural record*, and the body of this paper deliberately carries only the spine. Appendix A — the *Hagiography of the Exceptional Manifolds* — holds everything: the seed and the shared-form schema instantiated by every storey; all six regular polychora with full f-vectors, symmetry data, chromatic V_4 facet censuses, and Klein-address counts; the Weyl structures with their exponents and orphan couplers; the complete E_8, Leech, and Craig sections (node types, nucleus/void decompositions, holy censuses, phase tables); the cross-scale identities; and the homology of every object — 659 preserved data rows, each tiered (verified / characterized / open) and regenerated mechanically from the canonical sources. Appendix B carries the enumeration ledgers and the workbook index; Appendix C the laboratory compendium (a 37-laboratory curated snapshot of the 2026-07-04/05 arc — the full audited corpus is the 148 laboratories of §14 — including the step-one cross-parity construction of Theorem 4.3 and the independence audits behind Remarks 13.2 and 9.5, and the provenance trace of §14); Appendix D the reproduction commands, the artifact inventory, and the SHA-256 hashes of every immutable claim record.

2 The one-parameter engine

[Dynamic 1 of §1.6.]

Before any construction, the reader should hold the framework's actual shape: *one equation, one free choice, and a closed recursion.* Every constant in this paper is downstream of the twin-prime pair $(3,5)$, and the sections that follow are consequences, not postulates. This section states the compression theorem, the recursion that generates and terminates the tower, and the four independent proofs that the termination is structural.

2.1 The One Equation

Define, at any level n with twin pair $(p, q = p{+}2)$: the *Coxeter gauge* $h_C = 2pq$ (this section's h; the tower sections use the *conductor* $h = pq$ — the doubling is algebraically transparent, $(\mathbb{Z}/pq)^* \cong (\mathbb{Z}/2pq)^*$, and every formula below names its gauge), the *shell quantum* $\delta = 1/(2\sqrt{q})$, and $\varphi_q = (1+\sqrt{q})/2$ (the golden ratio φ at $q = 5$). The level-0 constants were not posited — they were *computed*: running the cross-parity projection of §4 on the 240 roots, the eigenplane data returns the projection shift $S = (5+2\sqrt{5})/10$ and the shell spacing $\delta = \sqrt{5}/10$ together, as outputs of one built construction (the projection-shift formula, proved). The identities below were then proved *about* those computed outputs; the first line of the proof is the non-trivial fact that the machine's own projection constant is a golden multiple of its own shell spacing.

Theorem 2.1 (Unification identity — the One Equation)**.** *At each level, with $S := \delta\,\varphi_q^3$,*

$$S^2 \;=\; \frac{p\,\varphi_q^6}{2h_C}.$$

Proof in three lines. *At level 0 the projection constant satisfies $\delta\varphi^3 = (\sqrt{5}/10)(2+\sqrt{5}) = (5+2\sqrt{5})/10 = S$ (golden scaling); $2h_C\,\delta^2 = p$ (shell-quantum–Coxeter coupling: $2\cdot 2pq/(4q) = p$);*

substitute: $S^2 = \delta^2\varphi_q^6 = (p/2h_C)\,\varphi_q^6$. $\square$ ***[EXACT]*** Provenance: *nothing here is a definition dressed as a theorem — S and δ are computed outputs of the built level-0 construction, the identities are proved of those outputs, and the equation is exact gated arithmetic at all three levels in both gauges. What remains open at levels 1–2 is an* enumeration*, not the equation: performing the cross-parity projection on the Leech and Craig lattices themselves, for which the equation already supplies the exact parameters it must return ($\delta_1 = 1/(2\sqrt{7})$, $S_1 = \delta_1\varphi_7^3$, and the level-2 pair)* ***[PREDICTED]****.*

The consequence is a five-to-one parameter collapse: before this identity the framework carried p, q, h_C, δ, S as separate constants; after it, *the only free choice in the entire program is which twin pair seeds it*. The kissing number itself is seed arithmetic, $K_0 = 2(p^2-1)\,pq = 2\cdot 8\cdot 15 = 240$, and the rank, populations, and clock periods follow (§6). The identity is gated exactly at all three levels, in both conductor gauges, in `information_thermodynamic_ledger` (13/13) with the gauge reconciliation carried by `refolding_experimental` (26/26; the suite that proves both conductor gauges yield the same exact identity), and its role-relativity across all seven tower windows in `tower_role_relativity` (C29).

2.2 The convexity sequence and the singularity cubic

Evaluating S along the tower gives the *convexity sequence*

$$S_0 = \tfrac{5+2\sqrt{5}}{10} \approx 0.947, \qquad S_1 \approx 1.145, \qquad S_2 \approx 1.693,$$

with the exact statements $S_0 < 1 < S_1 < S_2$ (all four comparisons decided by integer inequalities; C30). **Level 0 is the only level with $S < 1$:** the base fails to close by the convexity gap $1 - S_0 = \frac{1}{2} - \frac{\sqrt{5}}{5}$ ($\approx 5.3\%$, exact in $\mathbb{Q}(\sqrt{5})$), and that irreducible openness is the machine's standing store of potential — the higher storeys are over-closed (frozen). The boundary is itself algebraic:

Proposition 2.2 (Singularity cubic)**.** *$S(q) = 1$ if and only if $u = \sqrt{q}$ satisfies $u^3+3u^2-13u+1 = 0$, and this cubic has a root strictly between $\sqrt{5}$ and $\sqrt{7}$: the phase boundary between the dynamic and frozen regimes falls exactly between the seed's outer prime and its successor.* ***[EXACT]*** *(Derived, not quoted: expand $(1+u)^3 = 16u$; signs at $u = \sqrt{5}, \sqrt{7}$ decided by squaring.* `metallic_delta_family`*, 8/8.)*

The golden ratio is not an aesthetic choice. Within the metallic family $\delta_n^2 = 1/(4(n^2{+}4))$ the quantum $2h_C\,\delta_n^2 = 15/(n^2{+}4)$ (with $h_C = 30$ the level-0 Coxeter gauge as above) is a prime integer *only* at the golden member $n = 1$ (value 3, the inner twin prime; scanned exactly to $n = 1000$); the silver member carries the $\sqrt{2}$ secondary embedding of the quantum lift (§5.2), and the nickel member ($n = 5$, $q = 29$) returns the fraction 15/29 carried by σ_{29} — the cross-parity involution introduced in §4 — (C30).

2.3 The Möbius recursion, and four terminations

The tower is not a ladder; it is *one recursive structure*. The inner primes obey $p_{n+1}^2 = p_n^3 - p_n + 1$: from the seed, $3^3{-}3{+}1 = 25 = 5^2$ and $5^3{-}5{+}1 = 121 = 11^2$; at level 2 the recursion returns $11^3{-}11{+}1 = 1321$, which is *prime, not square* — there is no level 3, and the terminus wraps to the start: $1321 \bmod 13 = 8 = \dim E_8$. Unwrapped, the recursion presents as three tower perspectives chained by fold identifications (shared conductors 143 and 323 are the same component seen at two scales), with the outer closure $\varphi(899) = 840 = 8\cdot(3\cdot 5\cdot 7) = \text{rank} \times$ (all tower primes) connected through the *orphan prime* 7 — the one prime dividing the Weyl-group order but not the conductor, whose role the sieve below makes exact. Four independent facts close the tower **[EXACT]**:

1. **Arithmetic.** 1321 is prime (the recursion demands a square). The recursion is itself rigid: across polynomial degrees 2–5 (17,010 trials) the cubic from seed 3 is the *unique* all-prime ladder generator, and on the elliptic curve $y^2 = x^3 - x + 1$ (discriminant -368) the only integer points with both coordinates prime are $(3, 5)$ and $(5, 11)$ — the ladder, exactly.
2. **Clock.** The simple period formula breaks at the would-be $(17, 19)$ level: period 144, not the expected 660 — the desynchronization is the boundary.
3. **Span.** At each realized level the substrate's address arithmetic spans its full lattice (subgroup index 1 at 15/35/143); at the would-be continuations the span fails at index 2/3/6 — the arithmetic witness of finitude.
4. **Planes.** The invariant-plane ladder $4, 12, 60$ multiplies by the seed primes ($\times 3$, $\times 5$) and the continuation breaks the ratio pattern ($\times 2.4$) — the geometry stops where the arithmetic does.

The pair $(17, 19)$ ($h = 323$) is therefore not a fourth storey: it is the base of a *second crystal* — a parallel hierarchy whose L_0 it is, resonant with but not stacked on this one. (The discovery registry carries the stronger fourth-storey reading as a recorded retraction; the out-of-sample test this distinction enables is §17.3.)

2.4 Moments are dimensions; the sieve and its resurgence

Two further exact results locate the tower *inside* the six-shell data of §4.1, and belong up front because they show the one-parameter engine acting.

Theorem 2.3 (Moment–dimension identities). *Let $M_j = \sum_{roots} k^j$ over the shell indices. Then $M_2/M_0 = 4 = \mathrm{rank}(D_4)$ (the atom), $M_4/M_0 = 24 = \dim(Leech)$, and the variance is $M_4/M_0 - (M_2/M_0)^2 = 8 = \mathrm{rank}(E_8)$. The generating function is exact: $M_{2n}/M_0 = \frac{1}{3} \cdot 1^n + \frac{7}{15} \cdot 4^n + \frac{1}{5} \cdot 9^n$. The pattern* fails *at M_6 and M_8 — and the failure is structured: the excesses over the tower dimensions are* $176 - 120 = 8 \cdot 7$ *and* $1432 - 288 = 8 \cdot 143$ *(rank times orphan; rank times h_2). All odd moments vanish (the palindrome). The partition function at inverse temperature* $\ln\varphi$ *closes golden: $Z(\ln\varphi) = 64\varphi = \mathrm{rank}^2 \cdot \varphi$, exact in $\mathbb{Q}(\sqrt{5})$. **[EXACT]** (C31.)* Interpretive scope ***[READING]***: *the displayed arithmetic is exact;* reading M_2/M_0 as $\mathrm{rank}(D_4)$ *is an empirical match, while reading M_4/M_0 as* $\dim(\mathrm{Leech})$ *is the genuine tower prediction.*

The exponent spectrum itself is the output of the oldest algorithm in mathematics: an Eratosthenes sieve on $\{1, \ldots, 29\}$ by the primes of $h_C = 30$ has stage counts $29 \to 15 \to 10 \to 8$, the survivors are exactly the Coxeter exponents $\{1, 7, 11, 13, 17, 19, 23, 29\}$ (the integers governing the rotation spectrum of E_8's canonical element; defined with the root-system preliminaries in §3), the intermediate count 10 is the denominator of $\delta = \sqrt{5}/10$, and the sieved primes *resurface* as the shell-population factors ($24 = 8 \cdot 3$, $40 = 8 \cdot 5$) with the single orphan prime 7 exactly accommodated by the remaining shell ($56 = 8 \cdot 7$): one prime per shell pair, nothing left over. E_8 is cyclotomically saturated — rank $= \varphi(h)$, defect zero, uniquely among the exceptionals (E_6: 2, E_7: 1; the A-series prime-conductor degenerate case is disclosed, not hidden) — and the tower preserves that saturation by construction, $\dim_n = \varphi(h_n)$ (C32). The binomial seal $\binom{15}{4} = 1365 = 3 \cdot 5 \cdot 7 \cdot 13$ is itself one of the three σ-orphan primal sources of the divisor-flow topology (§8).

3 Mathematical preliminaries

We fix notation and recall standard structures; everything in this section is classical [7, 8, 9].

The E_8 root system. $\Phi(E_8)$ consists of 240 vectors in $\mathbb{R}^8$: the 112 *integer-type* roots $\pm e_i \pm e_j$ $(i < j)$, which form a D_8 subsystem, and the 128 *spinor-type* roots $\frac{1}{2}(\pm 1, \ldots, \pm 1)$ with an even number of minus signs (the half-spinor orbit S^+). All have norm^2 2. The Coxeter number is $h_C = 30$ with exponents $\{1, 7, 11, 13, 17, 19, 23, 29\}$.

The binary Golay code and the Leech lattice. The extended binary Golay code $G_{24} \subset \mathbb{F}_2^{24}$ is the unique $[24, 12, 8]$ code; its weight enumerator is {0:1, 8:759, 12:2576, 16:759, 24:1}; its weight-8 words are the 759 *octads* of the Steiner system $S(5, 8, 24)$. The Leech lattice Λ_{24} is the unique even unimodular lattice in dimension 24 with no norm-2 vectors; its minimal shell (norm 4) has 196,560 vectors and its automorphism group is the Conway group $\mathrm{Co}_0 = 2.\mathrm{Co}_1$.

Craig lattices. Craig's construction realizes dense lattices as ideal sublattices of cyclotomic rings $\mathbb{Z}[\zeta_p]$. Our level-2 object lives in $\mathbb{Q}(\zeta_{143})$, of degree $\varphi(143) = 120$.

The Klein quartic. The genus-3 curve $x^3y + y^3z + z^3x = 0$ attains the Hurwitz bound $|\mathrm{Aut}| = 84(g - 1) = 168$; $\mathrm{Aut} \cong PSL(2, 7)$, and its regular $\{7, 3\}$ map has 24 heptagonal faces, 84 edges, 56 vertices.

The rook graph. $\mathrm{SRG}(25, 8, 3, 2)$: vertices the cells of a 5×5 grid, adjacent when they share a row or column; spectrum $\{8^1, 3^8, (-2)^{16}\}$.

Braids and Burau. $B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$; the reduced Burau representation $\beta : B_3 \to GL_2(\mathbb{Z}[t, t^{-1}])$ is faithful; the writhe is the exponent sum.

CHSH. For dichotomic ± 1 observables A_0, A_1, B_0, B_1, $S = E(A_0B_0) - E(A_0B_1) + E(A_1B_0) + E(A_1B_1)$. Local deterministic (and hence local hidden-variable) models obey $|S| \leq 2$; quantum mechanics attains $|S| = 2\sqrt{2}$ (Tsirelson); the algebraic maximum is 4 [1, 2].

Landauer–Bennett. Erasing one bit dissipates at least $k_BT \ln 2$; computation can proceed with arbitrarily little dissipation if logically reversible, paying only at irreversible commits [3, 4].

Number-theoretic functions. φ is Euler's totient, σ the divisor sum, $\mathrm{ord}_h(2)$ the multiplicative order of 2 modulo h.

4 The cross-parity construction

[Dynamics 2–4 of §1.6: the projection, the golden fold, and the cancellation algebra.]

Everything shell-indexed in this paper stands on one construction, so the paper begins with it. Nothing in this section is asserted without a mechanism, and every clause is gated by exhaustive computation (Table 1, C01–C05).

4.1 The six-shell grammar

This subsection constructs the projection on which every shell-indexed quantity in the paper stands; nothing about it is asserted without a mechanism.

Definition 4.1 (Cross-parity projection)**.** *Let $w = s_1 \cdots s_8$ be a Coxeter element of $W(E_8)$, of order $h_C = 30$; its eigenplane exponents are the units $\{1, 7, 11, 13, 17, 19, 23, 29\}$ of $(\mathbb{Z}/30)^*$, in conjugate pairs $\{1, 29\}, \{7, 23\}, \{11, 19\}, \{13, 17\}$. Let τ be an* odd *coordinate mirror — a sign flip* $\mathrm{diag}(\pm 1, \ldots, \pm 1)$ *of determinant -1, the reference choice being $\tau_8 : e_8 \mapsto -e_8$. Such a mirror is* not *a symmetry of $\Phi(E_8)$: it preserves the integer-type roots but carries the half-spinor orbit S^+ onto the opposite orbit S^-, i.e. it maps E_8 onto its chirality twin. The* cross-parity element *is the conjugate $w_\tau = \tau w \tau$ — the twin's Coxeter rotation, viewed in the original frame — and the* cross-parity projection *sends each root to its component in the 4-plane spanned by the $\{1, 29\}$ and $\{11, 19\}$ eigenplanes of w_τ. The name is now literal: the roots of one parity class are read in the Coxeter eigenframe of the* other *parity class.*

Theorem 4.2 (The six-shell law)**.** *Under the cross-parity projection the 240 roots take exactly six squared radii,*

$$r_k^2 = 1 + k\delta, \qquad \delta = \frac{\sqrt{5}}{10}, \qquad k \in \{-3, -2, -1, +1, +2, +3\},$$

with populations $[24, 56, 40, 40, 56, 24]$, and with perfect type separation: the $|k| = 2$ shells contain only integer-type (D_8) roots, the $|k| \in \{1, 3\}$ shells only spinor-type (S^+) roots. The $k = 0$ layer is empty. Moreover $\delta^2 = 1/20$ and $2h_C\delta^2 = 3$ exactly in $\mathbb{Q}(\sqrt{5})$. ***[EXACT]***

Proof method. Exhaustive exact computation over the 240 roots: the projection is a $\mathbb{Q}(\sqrt{5})$-linear map; each squared radius is evaluated in $\mathbb{Q}(\sqrt{5})$ and matched against $1+k\delta$; populations and types are counted. The census is reproduced independently from three constructions of $\Phi(E_8)$ (integer model, icosian model, Elser–Sloane model). The palindrome is forced structurally by Theorem 5.6 below. □

The definition is not one choice among many; the mechanism is exhaustively characterized:

Theorem 4.3 (The parity switch and the selection rule)**.** *(i) Over* all 256 *coordinate sign flips τ: every odd flip ($\det \tau = -1$; 128 cases) yields the six-shell law of Theorem 4.2 with perfect type separation, and every even flip ($\det \tau = +1$; 128 cases, including the identity — the standard Coxeter plane) yields* no *type separation. Separation exists only across the parity boundary: one must read the lattice in its mirror twin's frame. (ii) Of the six ways to pair the four exponent pairs into a 4-plane, exactly two produce the separation — the exponent sets $\{1, 29, 11, 19\}$ and $\{7, 23, 13, 17\}$, precisely the two closed under the Galois action of $\mathbb{Q}(\zeta_{30})$ fixing $\sqrt{5}$; the four Galois-broken pairings mix types. (iii) The mechanism is E_8-special: it requires the integer-type norm^2 2 to coincide with the spinor-type norm^2 $d/4$, which holds only at $d = 8$ (for D_{16}^+ the spinor norm is 4, for the Niemeier frames 6), so the construction cannot even be posed one storey up.* ***[EXACT]*** *(exhaustive gated censuses: 256 mirrors × 240 roots, and all six pairings).*

Three remarks fix the reading. First, the type separation is a *parity* phenomenon (i) *and* an *arithmetic* one (ii): the mirror supplies the crossing, the Galois closure selects the plane pair — both are needed, neither is a numerical accident. Second, the reference mirror τ_8 is not scaffolding to be discarded after use: the same operator returns in §7 as the involution whose fixed manifold is the 84-root anchor set (Theorem 7.1) — the operator that *creates* the readout frame is the operator whose rigidity the ledger prices. Third, the projection is $\mathbb{Q}(\sqrt{5})$-exact: the golden step $\delta = \sqrt{5}/10$ of the shell law is the trace field of the 30-fold rotation, which is why the shells register epitaxially (Proposition 8.2).

Theorem 4.4 (Lattice emergence of the metric)**.** *The $k = +3$ shell of the projection is a 24-cell. Intrinsically (as the minimal vectors of D_4) its normalized Gram matrix takes four rational values.*

As a projected figure from E_8, *over all* $\binom{24}{2} = 276$ *pairs, each 8D inner-product class maps to exactly one emergent 4D Gram value in* $\mathbb{Q}(\sqrt{5})$ *— five values, zero exceptions:*

$$\text{ip}\,{-2} \mapsto -1, \quad \text{ip}\,{\mp 1} \mapsto \frac{\mp 4 \mp \sqrt{5}}{11}, \quad \text{ip}\,0 \mapsto \pm\frac{3 - 2\sqrt{5}}{11},$$

where the two ip 0 *values are selected by the* sign correlation of the eighth coordinate *— the very coordinate the cross-parity mirror flips — and the Galois involution* $\sqrt{5} \mapsto -\sqrt{5}$ swaps *the two* ip 0 *classes.* ***[EXACT]*** *(exhaustive: 276/276).*

Three consequences. First, *the projection extends the number field*: metric structure invisible in D_4 (where $K = \mathbb{Q}$) emerges from the ambient lattice ($K = \mathbb{Q}(\sqrt{5})$) — geometry is downstream of the lattice arithmetic, and the emergent labels (type, shell, sign encoding) carry strictly positive information the free 24-cell lacks. Second, this is *how the Galois generator acts on the substrate*: not as a pointwise vector mirror but on the *emergent* metric structure, swapping conjugate Gram classes — the realization that the involution registers of the running system carry (Remark 9.5). Third, the palindrome $p(k) = p(-k)$ follows from the field automorphism alone (σ maps $r_k^2 \mapsto r_{-k}^2$ and preserves algebraic multiplicities) — the self-duality forcing of Theorem 5.6 and the Galois forcing are two faces of one fact.

The six shells form a *discrete grammar of boundary access* (the physics reading of this grammar is developed in §15): before constraints, six labels admit an ambient S_6 permutation grammar; parity ($k \mapsto -k$), radius, and type constraints break it into three conjugate shell-pair orbits $\{\pm 1\}, \{\pm 2\}, \{\pm 3\}$. The empty center is the observer's seat: the one radius the grammar *names* but never populates — the missing frame (§16).

4.2 V_4 at the standing nodes; the 24-cell as mediator

Proposition 4.5 (V_4 as Galois polarization)**.** *The group* $V_4 = \{1, \sigma_{29}, \sigma_6, \sigma_{34}\}$ *of units* $\{1, 29, 6, 34\}$ mod 35*, each self-inverse, is canonically* $\mathrm{Gal}(\mathbb{Q}(\sqrt{5}, \sqrt{7})/\mathbb{Q}) \cong (\mathbb{Z}/2)^2$*:* σ_{29} *negates* $\sqrt{5}$*,* σ_6 *negates* $\sqrt{7}$*,* σ_{34} *negates both. Its action on polarization state is exactly the Stokes Klein-four* $\{(I, Q, U, V) \mapsto (I, \pm Q, \pm U, V)\}$*: two orthogonal half-wave plates.* ***[EXACT]*** *(group structure); the optical identification is the gauge census of §10.*

Proof. $29 \equiv -1 \bmod 5$ and $29 \equiv 1 \bmod 7$, so σ_{29} is $(-, +)$ under the CRT splitting $(\mathbb{Z}/35)^* \cong (\mathbb{Z}/5)^* \times (\mathbb{Z}/7)^*$; similarly 6 is $(+, -)$ and $34 = 29 \cdot 6 \bmod 35$ is $(-, -)$. Each squares to 1 mod 35. The identification with quadratic-field conjugations follows since $\mathbb{Q}(\zeta_5)$ contains $\sqrt{5}$ and $\mathbb{Q}(\zeta_7)$ contains $\sqrt{-7}$ (real subfield: $\sqrt{7}$ conventions as used throughout). □

Physically **[READING]**: where two chiral boundary streams (forward and backward waves) cancel at standing-wave nodes, the residue algebra is exactly this V_4 — four fixed polarization outcomes, not a continuum. The *24-cell* — the unique self-dual regular 4-polytope, with f-vector $(24, 96, 96, 24)$ — is the involutive boundary mediator: its self-duality forces the palindromic populations (Theorem 5.6), and its empty center seats the observer.

Five optical experiments (cavity–aperture mutual information staircase; Mie-resonance alignment with r_k; V_4 polarization classes at cavity nodes; inverse-scattering recovery of the shell census only; closure-pressure rendering) constitute the physical falsification program **[OPEN]**.

4.3 The golden fold, exactly charted

A second lane reads the same 240 roots through the Elser–Sloane projection $\varepsilon(x) = (x_1 + \varphi x_5,\ x_2 + \varphi x_6,\ x_3 + \varphi x_7,\ x_4$ into $\mathbb{Q}(\varphi)^4$ — the $E_8 \to H_4$ *golden fold*, whose integral points $\mathbb{Z}[\varphi]^4$ are the coordinate shadow of the *icosians* (the golden-integer quaternions whose 120 units are the 600-cell's vertices; this is the "icosian model" invoked in the proof of Theorem 4.2). Its census is exact (`golden_join_materialization`, 7/7; C34):

Theorem 4.6 (Type is the lock bit)**.** *A root is integer-locked in $\mathbb{Z}[\varphi]^4$ under ε if and only if it is a D_8 (integer-type) root:* 112/112 *locked,* 0/128 *of the spinor roots (each spinor image has a half-integer golden component). The cross-parity free type bit of Theorem 4.3's lane and the icosian lock bit are the* same bit read on two constructions*. All 84 anchors lock; the 24 inner anchors have zero golden component (the pure rational sublattice); the 60 outer anchors split* 48+12 *by unit golden-slot count.* ***[EXACT]***

Theorem 4.7 (Five golden norms; the rational nucleus)**.** *The icosian norm $N(\varepsilon(r))$ takes exactly five values,*

$$\{\,\varphi^{-2},\ 2,\ 2+\varphi,\ 2\varphi^{2},\ \varphi^{4}\,\}\quad\textit{with populations}\quad [24,\,24,\,144,\,24,\,24],$$

the rational part always 2 *(the E_8 norm survives the fold; only the golden part varies). The Galois-rational class ($N = 2$ exactly) is* precisely *the inner 24-cell: the achiral nucleus needs no label — it is the fold's rational core. The two extreme classes (φ^{-2} and φ^4, 24 roots each) have identical anchor composition — a mirror pairing at the ends of the golden ladder.* ***[EXACT]***

The fold also *sharpens the program's standing open.* The observer rail (§14) is census-identical to the coordinate rail but root-by-root different; the natural conjecture was that the golden fold mediates them. The join test answers exactly: over the eight (norm-class × lock) keys, *zero* determine the observer columns — the observer rail provably does not factor through the fold invariants. If a golden join exists it is a root-by-root correspondence (an explicit matching in the H_4 orbit), not an invariant relabel. The open survives, sharper **[OPEN]**.

5 The dualistic minimum

[Dynamic 5 of §1.6.]

5.1 The relational prime

Principle 5.1 (Relational Prime)**.** *A singleton cannot express relation: comparison, distance, symmetry, chirality, cause, observer/object separation are all undefined for one undifferentiated entity. The integer 2 is the minimal relational primitive; unity is pre-relational process potential.* ***[READING]***

The principle earns its place by its exact consequences: the quantum boundary it forces (§5.2) and the palindrome family its self-dual folding generates (§5.3).

5.2 The diagonal lift and the exact quantum closure

A two-state system defines an edge of length 2 (the classical contrast scale), a square of area $2 \times 2 = 4$ (the unconstrained algebraic table), and a diagonal $2\sqrt{2}$ (the self-dual closure across the square). This geometry is realized exactly by the CHSH functional.

Proposition 5.2 (Classical edge)**.** *Over all* 16 *deterministic assignments $A_0, A_1, B_0, B_1 \in \{\pm 1\}$, $\max|S| = 2$.* ***[EXACT]***

Proof. $S = A_0(B_0 - B_1) + A_1(B_0 + B_1)$; one of $B_0 \mp B_1$ is 0 and the other ± 2, so $|S| = 2$ for every assignment. (Exhaustive census confirms.) $\square$

Theorem 5.3 (Exact symbolic closure)**.** *For the singlet-frame correlators $E(a, b) = \cos 2(a - b)$ at the canonical settings $a_0 = 0$, $a_1 = \pi/4$, $b_0 = \pi/8$, $b_1 = 3\pi/8$, each correlator lies in $\mathbb{Q}[\sqrt{2}]$:*

$$E_{00} = E_{10} = E_{11} = \tfrac{\sqrt{2}}{2}, \qquad E_{01} = -\tfrac{\sqrt{2}}{2},$$

and the CHSH functional closes exactly at

$$S \;=\; E_{00} - E_{01} + E_{10} + E_{11} \;=\; 4 \cdot \tfrac{\sqrt{2}}{2} \;=\; 0 + 2\sqrt{2} \;\in\; \mathbb{Q}[\sqrt{2}].$$

[EXACT] *(field arithmetic on $\mathbb{Q}[\sqrt{2}]$ pairs; no sampling, no floating point).*

Remark 5.4. *This computation is standard quantum mechanics; the paper adds nothing to Bell–Tsirelson theory and claims nothing from it. The content here is methodological: the exact lane is kept algebraic, the engine's discretized audit gap is disclosed rather than patched, and sampling is never allowed to stand in for proof. The substrate-wide reading of the ladder is Conjecture 18.3, not this theorem.*

Corollary 5.5 (Tsirelson factorization)**.** $S_q^2 = (2\sqrt{2})^2 = 8 = 2 \cdot 4 = S_c \cdot S_a$*:* the quantum closure is the geometric mean of the classical edge and the algebraic envelope. *Equivalently the ladder $2 \to 2\sqrt{2} \to 4$ is multiplication by the unit diagonal $\sqrt{2}$ at each rung, terminating at $4 = |V_4|$, the two-bit square envelope. We record the factorization as arithmetic structure; no further structural content is claimed for it here.* ***[EXACT]***

Three evidence lanes are kept separate throughout the program and never collapsed: a legacy engine's orbit-discretized audit, which reports the classical edge $S = 2.0$ and is retained as a disclosed gap; the exact symbolic lane of Theorem 5.3, which is the proof; and finite-shot sampling support ($S \approx 2.83$ at 2×10^4 shots/setting) **[MEASURED]**, which is corroboration, never proof. The proposal that this same ladder organizes *every* layer of the substrate (gauge envelope, information seam, commit) is Conjecture 18.3 (§18) — deliberately not assumed here.

5.3 The palindrome family

Self-duality is the dualistic minimum folded onto itself: an object identical with its own dual. The substrate's palindromes are the visible signature.

Theorem 5.6 (Self-duality forces the palindrome)**.** *The 24-cell is the unique self-dual regular 4-polytope, and any cross-parity-type projection commuting with its central involution carries mirror shells to mirror shells; hence the shell populations must satisfy $p(k) = p(-k)$. The measured* $[24, 56, 40, 40, 56, 24]$ *realizes the forced palindrome.* ***[EXACT]*** *(census); the forcing chain is structural.*

Theorem 5.7 (Homological palindrome)**.** *Over $GF(2)$, the boundary maps of the 24-cell's face lattice (computed from the explicit vertex set $\{\pm e_i \pm e_j\}$, all 3-cliques as triangles, 24 octahedral cells located by the dual 24-cell) have ranks* $(23, 73, 23)$ *— palindromic — with Betti numbers* $(1, 0, 0, 1)$*, i.e. the 3-sphere, and $\chi = 0$; the central rank satisfies* $73 = 96 - 23$*, forced by the f-vector. Dually paired polychora reverse their rank triples:* $(15, 17, 7) \leftrightarrow (7, 17, 15)$ *for the 8/16-cell pair,* $(119, 601, 599) \leftrightarrow (599, 601, 119)$ *for the 600/120-cell pair.* ***[EXACT]***

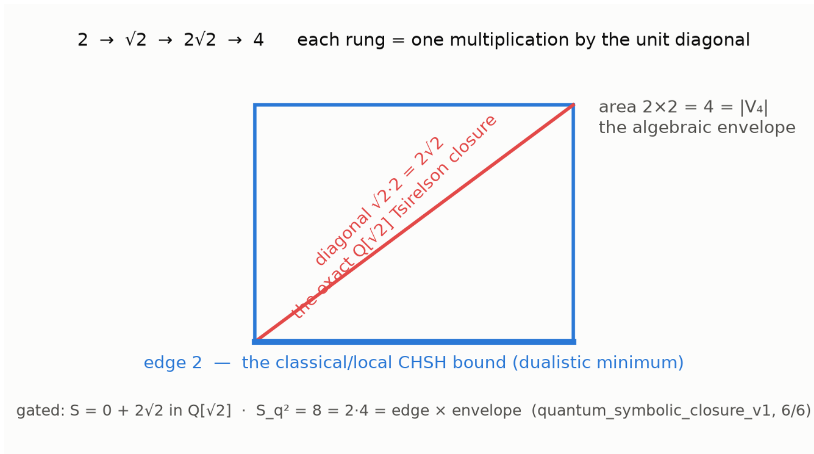

Figure 1: The diagonal lift: the classical edge 2, the exact $\mathbb{Q}[\sqrt{2}]$ closure $2\sqrt{2}$, the envelope $4 = |V_4|$ (Theorem 5.3, Corollary 5.5).

Proposition 5.8 (Twin-frame law)**.** *For twin primes $p, q = p + 2$ with midpoint $f = p + 1$ and conductor $h = pq = f^2 - 1$:*

$$\sigma(h) = (p{+}1)(q{+}1) = f^2 + 2f, \qquad \varphi(h) = (p{-}1)(q{-}1) = f^2 - 2f, \qquad \sigma - \varphi = 4f.$$

[EXACT] *(immediate; verified bit-exact on all 35 twin conductors below 10^3).*

Theorem 5.9 (Eversion)**.** *Let $M = \times(-\zeta_h)$ be the characteristic isometry at conductor h (order $2h$, characteristic polynomial Φ_{2h}). Every chirality residue k is a unit mod $2h$, hence odd; therefore on every invariant rotation plane, M^h is rotation by $k/2$ turn = an odd multiple of $180°$, i.e.*

$$M^h = -I \quad \textit{simultaneously on all } \varphi(h)/2 \textit{ planes.}$$

The eversion — the global antipodal inside-out — is thus a single, simultaneous, half-period act at every level (M^{15}, M^{35}, M^{143} of orders 30, 70, 286). ***[EXACT]***

Two chirality objects must never be conflated, and the program names them apart **[EXACT]**: *simple chirality* is the isoclinic ± 1 sign of one plane pair — real, but a shadow — while *hyper-chirality* is the full per-plane magnitude spectrum: at level n the characteristic isometry carries $\varphi(h_n)/2 = \{4, 12, 60\}$ independent invariant planes, each with its own angle and sign, quantum $1/(2h_n)$ turn per level ($12°$ at E_8 — with plane magnitudes $12° \cdot \{1, 7, 11, 13\}$, of which the H_4 Coxeter clock keeps only $\{12°, 132°\}$: the orphan-7 and bridge-13 magnitudes are non-Coxeter in the 600-cell group, unified only in E_8). The plane count multiplies by the seed primes up the tower ($\times 3$, $\times 5$), and chirality is measured on at least eight independent channels, never collapsed to one axis (`hyper_chirality_leech_craig` 7/7; `chirality_multichannel_spectrum` 7/7).

Time itself is palindromic in this substrate **[READING]**: the forced 12-tick period (the 4D chiral termination: $\binom{4}{2} = 6$ rotation planes $\times$ 2 senses) makes the neighborhood of any observation point a temporal palindrome, with the present as its axis of symmetry — a testable prediction (§16).

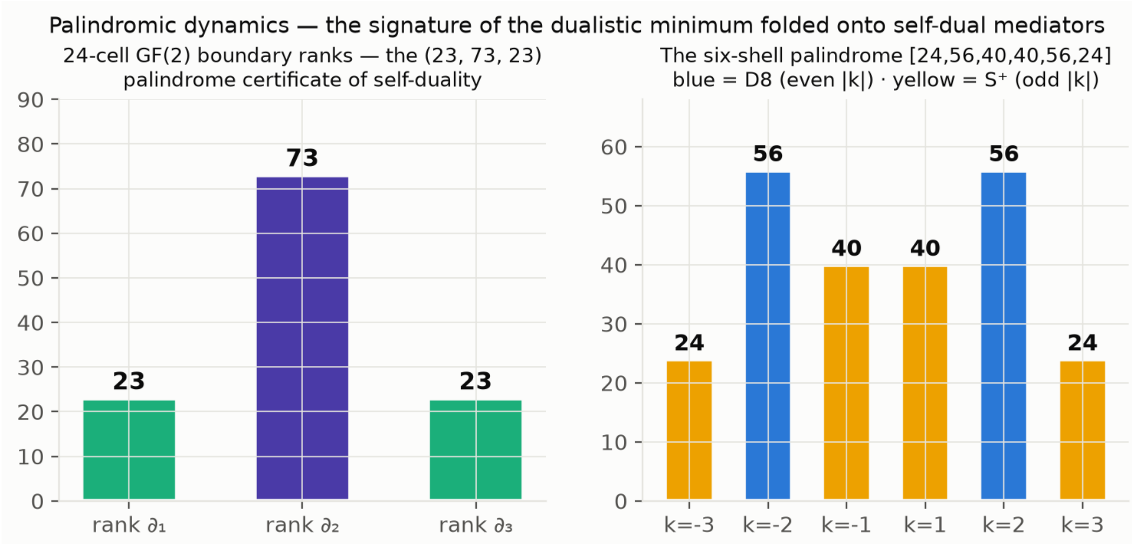

Figure 2: Palindromic certificates: the 24-cell ∂-rank palindrome $(23, 73, 23)$ and the six-shell palindrome (Theorems 5.6, 5.7).

6 The one-seed tower and the ZTC phylum

6.1 The tower

Theorem 6.1 (Frame–conductor identities). *With $f_n = p_n+1$ for the twin pairs $(3,5), (5,7), (11,13)$:*

$$h_n = f_n^2 - 1 = p_n q_n \in \{15, 35, 143\}, \qquad \varphi(h_n) = \{8, 24, 120\}, \qquad \sigma(h_n) = \{24, 48, 168\},$$

$$\mathrm{ord}_{h_n}(2) = \{4, 12, 60\}, \qquad \varphi(2h_n) = \{8, 24, 120\}, \qquad \varphi(h_n)/2 = \{4, 12, 60\}.$$

*The dimension ladder $8 \cdot 24 \cdot 120$ equals both the ambient dimensions of E_8, Leech, Craig and the vertex counts of their carrier polytopes (16-cell, 24-cell, 600-cell); the clock periods $4/12/60$ generate the 240-tick orbit $4{\cdot}\mathrm{lcm}(4, 12, 60)$; the doubled conductors $2h = 30/70/286$ are the isometry orders of Theorem 5.9, with $\varphi(2h) = \dim$ at every level. **[EXACT]***

No quantity above is chosen; each is φ, σ, ord, or an explicit construction over the seed. The tower's closure is not a hypothesis: the Möbius recursion terminates at the prime $1321 = 11^3 - 11 + 1$ with the wrap $1321 \bmod 13 = 8 = \dim E_8$, and the termination carries four independent proofs (§2). The pair $(17, 19)$ ($h = 323$) is the base of a second, parallel crystal — not a fourth storey of this one.

6.2 The phylum body plan

The program comprises a *family* of exact calculation systems — toral, tower, senary-division, kirigami, polarization, and cyclonic calculi — sharing four diagnostic characters **[EXACT]**:

1. **Exactness.** State is exact (integers, residues, $\mathbb{Q}(\sqrt{5})$ couples, finite-group elements), never floating point. A float merge is a logical erasure; only over exact state is cost countable.
2. **The seed.** No member has a free parameter the twin-prime seed does not fix.
3. **The tower.** Every member is indexed by $\{15, 35, 143\}$ and scale-covariant along the tower.

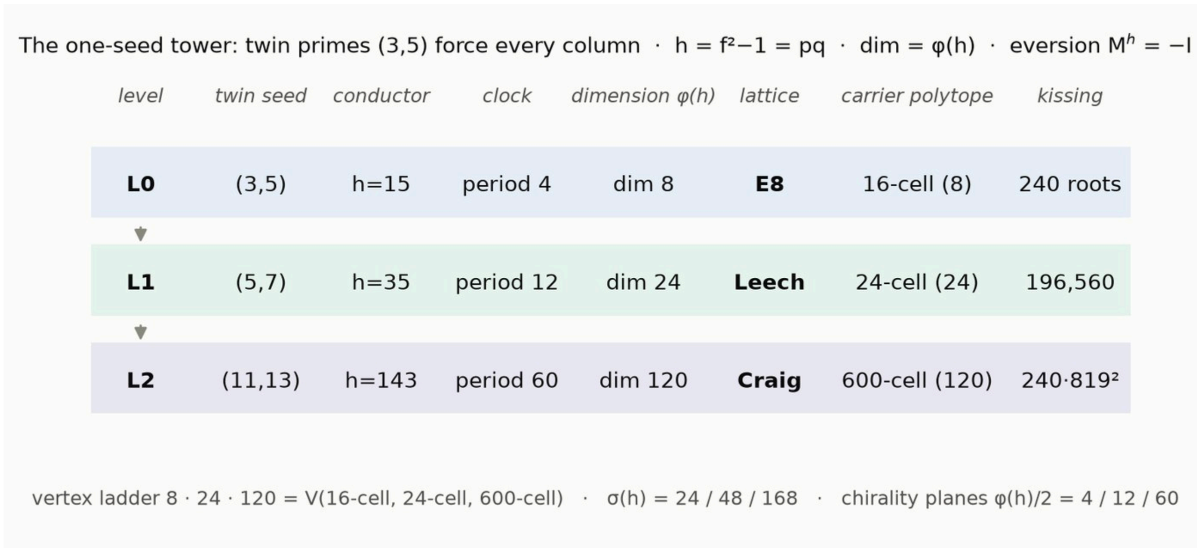

Figure 3: The one-seed tower (Theorem 6.1).

4. **The commutation spine.** Every member's operations split into an *order-blind measurement* (the abelian polarization register) and an *order-sensitive memory* (the non-abelian braid/isometry current), the difference priced by the 84-bit ledger of §9.1. (39 machine gates.)

6.3 The hyperbinary operator algebra

The substrate's mutation fabric deserves careful introduction: it is neither binary nor a loose "multi-valued logic," but a specific layered operand structure we call *hyperbinary*.

Definition 6.2 (The Lips_8 group)**.** $\mathrm{Lips}_8 = V_4 \times \langle F \rangle \cong (\mathbb{Z}/2)^3$ *is the group of eight operator words*

$$\{E,\ S_{29},\ S_6,\ S_{34},\ F,\ FS_{29},\ FS_6,\ FS_{34}\},$$

where V_4 is the Galois polarization group of Proposition 4.5 and F is the Möbius fold (the half-twist of the toral address). Every element is self-inverse. The anti-bus *involution pairs each word with its fold partner, $E \leftrightarrow F$, $S_{29} \leftrightarrow FS_{29}$, $S_6 \leftrightarrow FS_6$, $S_{34} \leftrightarrow FS_{34}$; applying it twice is the identity.* ***[EXACT]***

Definition 6.3 (Hyperbinary atoms)**.** *Each* Lips_8 *word decomposes into three binary* atoms*: the σ_{29} axis ($\sqrt{5}$ cancellation — the shell/type axis), the σ_6 axis ($\sqrt{7}$ cancellation — the context/boundary axis), and the fold bit (the anti-bus axis). The eight words are exactly the 2^3 atom combinations. Orthogonally to the atom* bits, *each atom carries a* ternary reliability *annotation in $\{-1, 0, +1\}$ (committed-low / undecided / committed-high) — measurement-coherence metadata stored beside the atom, never inside it.*

Proposition 6.4 (Atom–shell bridge)**.** *Two* Lips_8 *molecules of three atoms each give six atoms, mapping bijectively onto the six shells of Theorem 4.2: molecule 1 $(\sigma_{29}, \sigma_6, F) \mapsto (k{=}-3\ S^+,\ k{=}-2\ D_8,\ k{=}-1\ S^+)$ with phase roles (reverse-generative, structural order, observer binding); molecule 2 $\mapsto (k{=}+1, +2, +3)$ with roles (intention, formal anchor, forward synthesis). The operator fabric and the boundary grammar are the same six-fold object read twice.* ***[EXACT]***

Definition 6.5 (The seven-layer semantic stack)**.** *Every semantic event in the substrate is simultaneously addressable at seven layers:*

layer	*radix*	*object*	*values*
1	*binary*	*atom bit*	$0/1$
2	*ternary*	*reliability*	$-1/0/+1$
3	*senary*	*shell lane*	$k \in \{\pm 1, \pm 2, \pm 3\}$
4	*quaternary*	V_4 *cancellation*	$\{1, 29, 6, 34\}$
5	*septenary*	*Klein channel*	$0..6$
6	*sexagesimal*	*relation bucket / orbit hour*	$0..59$
7	*25-ary*	*void perspective*	$0..24$

The seven Klein channels $K_0..K_6$ (layer 5) rotate about a hidden eighth hub *— the void/transport axis — and the spoke census $7 \times 4 = 28$ recovers the D28 active-root block.* ***[EXACT]***

Remark 6.6 (The mutation membrane)**.** *Every state mutation in the architecture flows through exactly one* Lips_8 *word — no other mutation path exists. This single rule is what makes the machine enumerable, replayable, and falsifiable: any event stream can be checked for closure (all words in the canonical eight), collapsed along the anti-bus to verify reversibility, and re-emerged deterministically from its seed. The hyperbinary fabric is thus not a data format but the substrate's* grammar of change*; its seven simultaneous radices are why we call the substrate's operand structure hyperbinary rather than binary.*

7 The holy enumerations

[Dynamic 6 of §1.6: the full enumerations.]

7.1 E_8 and the anchor stratification

Theorem 7.1 (Anchor stratification)**.** *The cross-parity mirror τ_8 of Definition 4.1 acts on $\Phi(E_8)$ in three ways at once: it* fixes *pointwise exactly $84 = 24 + 60$ roots — the A24 locked core (a D_4 subsystem, the Hurwitz 24-cell) and the F60 free anchors, together the root set of the D_7 sublattice orthogonal to the mirror axis; it* permutes *the remaining $28 = \binom{8}{2}$ integer-type roots (D28, the block touching the mirrored coordinate) within the system; and it* everts *all 128 spinor roots onto the chirality twin ($S^+ \to S^-$, outside Φ). The four-class partition $24 + 60 + 28 + 128 = 240$ is thus read off a single operator — fixed / moved-within / everted — and coincides, root for root, with the nested-support geometry $D_4 \subset D_7 \subset D_8$ (+) spinors (audited 240/240, Theorem 13.1(i)). The substrate's gauge carries these four classes as its V_4 states $\{0, 1, 2, 3\}$, one-hot.* ***[EXACT]***

The realization must be stated with care, because two different objects answer to the same sentence. "The involution fixes the 84 anchors" is a theorem about τ_8 — the same mirror that creates the cross-parity frame (Theorem 4.3). The runtime's involution *register* matches every census (fixed count exactly 84) in its own constructed frame; the two-rail accounting is Remark 9.5.

The H_4 bridge: where the channels live in the Lie structure. The anchor stratification is not free-floating combinatorics; it sits inside the exceptional fold $E_8 \to H_4$. Under the golden projection the 240 roots cover the 120 vertices of the 600-cell exactly twice (the doubling law of §8), and the substrate's *H-type channels are the H_4 Weyl-orbit structure read back inside E_8*: the canonical per-root bridge census is 24 ($H_4{\cdot}D_4$ fixed 24-cell bridge) + 60 (anchor horizon)

+ 28 (active axis-7) + 32 (S^+ outer) + 96 ($H96$ golden-vertex center) = 240, with the bridge-register center $144 = 48_{D_8} + 96_{S^+}$ (a column census of the observer-rail register, §14). The 25-cell constellation beneath the router is likewise constructed, not chosen. The constellation itself — the five disjoint 24-cells tiling the 600-cell and the full family of 25 inscribed 24-cells — is *Moxness's construction* [13], from his $E_8 \to H_4$ development of the 600-cell; this paper re-derives it in exact arithmetic, and every use of it above the E_8/600-cell level — the router, the void-observer frame, the Leech and Craig carriers — is this program's application of the base, stated as such. The 600-cell contains exactly 25 inscribed 24-cells — the cosets of the five conjugate binary-tetrahedral subgroups of the icosian group $2I$, built in $\mathbb{Z}[\varphi_g]$ — whose exact pairwise vertex-overlap law is $\{0, 6, 24\}$ (100 disjoint pairs, 200 facet-sharing pairs); the *disjointness* graph is the 5×5 rook graph SRG$(25, 8, 3, 2)$, and by Shrikhande's theorem [18] the parameters $(25, 8, 3, 2)$ determine this graph *uniquely* (the $L_2(n)$ family admits an exceptional twin only at $n = 4$). The constellation's topology is therefore forced: given 25 objects with this regularity, the rook routing surface is the only possibility. **[EXACT]** (icosian construction, overlap law, and SRG parameters exhaustively verified; uniqueness classical).

Three layers close over this geometry, and they are consistent by construction, not by coincidence: the *static substrate* (the unique rook topology above, with $\beta_1 = 76/176$, Theorem 8.6); the *nucleus* (one of the 25 cells held empty as the active void — the observer's seat of §16); and *observer-relative routing* (the $625 = 25 + 500 + 100$ void grid, in which each nucleus splits its 8 disjoint cells into 4 same-row channel cells and 4 same-column complement cells while the 16 facet-sharing cells join the channel — a verified refinement of the static geometry through the nucleus, never a rival to it). The router of §13 is the third layer; its by-construction one-hot property is that refinement made addressable.

7.2 The Leech shell, materialized from a verified Golay code

The shape decomposition below and the weight enumerator are classical (Conway–Sloane [7]); what this section contributes is their *from-scratch, machine-gated* reconstruction — the Golay code built from its generator polynomial and verified term-by-term, then the shell materialized vector-by-vector — so that every downstream census in this paper rests on artifacts constructed inside the same exact discipline, plus the block-activity law of Theorem 7.3, which to our knowledge is new. The Golay code is constructed, not imported: the length-23 quadratic-residue cyclic code with generator polynomial $g(x) = x^{11} + x^{10} + x^6 + x^5 + x^4 + x^2 + 1$, extended by a parity bit; the resulting $[24, 12, 8]$ code's weight enumerator is verified term-by-term (759 octads enumerated explicitly). On that code:

Theorem 7.2 (Shape enumeration of the kissing shell)**.** *The* 196,560 *minimal vectors of* Λ_{24} *(norm*^{2} *32 in the scaled integer model) decompose exactly by shape:*

$$\underbrace{759 \cdot 2^7 = 97{,}152}_{(\pm 2)^8 \text{ on an octad, even minus}} + \underbrace{276 \cdot 4 = 1{,}104}_{(\pm 4)^2 \text{ on a pair}} + \underbrace{24 \cdot 2^{12} = 98{,}304}_{(\mp 3, (\pm 1)^{23}) \text{ Golay signs}} = 196{,}560,$$

with the cross-denominations $196{,}560 = 819 \cdot 240 = 1365 \cdot 144 = 13 \cdot 15{,}120 = 720 \cdot 273$. ***[EXACT]***

The epitaxial lift factor $819 = 3^2 \cdot 7 \cdot 13$ has a further exact decomposition, read off the per-vector Galois grammar. Grouping the 196,560 by their $\sqrt{7}$-axis profile marginal gives exactly nine classes, in three Galois-conjugate triples, with populations $240 \cdot \{196, 49, 28\}$ each; hence

$$819 = 3 \cdot (14^2 + 7^2 + 28),$$

the squares of the level-1 frame $14 = 2 \cdot 7$ and the orphan prime 7, plus $28 = \binom{8}{2}$, the E_8 rotation-plane count. And the joint ($\sqrt{5}$-charge) $\times$ ($\sqrt{7}$-pattern) census populates exactly $27 = 3^3$ cells — the level-1 analogue of the six-shell grammar is a two-axis, senary-cubed structure, V_4-invariant and serving as the apex's error syndrome (§9.6). **[EXACT]** (`leech_L1_galois_grammar` 8/8)

Theorem 7.3 (Holy categories and the block-activity law)**.** *Under the holy frames the shell splits as* $A + B + C = 720 + 90{,}720 + 105{,}120$*; and under an* E_8^3 *Golay-trio frame the census by block activity is*

$$\#\{\textit{exactly } k \textit{ blocks active}\} \;=\; 45 \cdot 16^k, \qquad k = 1, 2, 3 \;\longrightarrow\; 720,\; 11{,}520,\; 184{,}320,$$

with $16 = [E_8 : \sqrt{2}E_8]$ *the per-block glue index and* $16^3 = 4096 = |\mathbb{F}_2^{12}|$*. Derived independently by octad-trio combinatorics (*$3+84+672$ *trio patterns) and by explicit vector census; invariant across distinct trios. The category sums are classical in content; the block-activity enumeration* $45 \cdot 16^k$ *is, to our knowledge, new (no prior statement located in [7] or the Turyn/Lepowsky–Meurman* E_8^3 *literature [10]).* ***[EXACT]***

The stray prime of category C, $105{,}120 = 2^5 \cdot 3^2 \cdot 5 \cdot 73$, is numerically the 24-cell's cycle rank $\beta_1 = 73$ (Theorem 8.6); we record the coincidence as a cross-denotation **[READING]** and do not assert identity.

The full shell, three ways — and beyond the first shell

The enumeration of the kissing shell is not merely counted: it is *materialized* (every one of the 196,560 vectors built as exact integers of norm^2 32, with per-vector class labels), and it carries **three coexisting per-vector partitions**, each gated, agreeing exactly on the 720-vector core and nowhere else **[EXACT]**:

partition	classes	counts	status
geometric (block activity)	$k = 1, 2, 3$ blocks	720 / 11,520 / 184,320	gated (Theorem 7.3)
holy (Turyn frame)	A / B / C	720 / 90,720 / 105,120	A exact; B/C listing open
$\sqrt{5}$-type (Galois)	$t_5 = -4$ / 0 / $+4$	37,800 / 120,960 / 37,800	gated (17 checks)

The third row is the level-1 analogue of E_8's free type bit, and deserves statement. Realizing Λ_{24} as a rank-one module over the cyclotomic integers $\mathbb{Z}[\zeta_{35}]$ (via an explicit order-35 isometry g of the Gram form, so $\zeta_{35} \mapsto g$), the element $\sqrt{5} = \zeta_5 - \zeta_5^2 - \zeta_5^3 + \zeta_5^4$ becomes the integer operator $M_5 = g^7 - g^{14} - g^{21} + g^{28}$ with $M_5^2 = 5I$, and every minimal vector carries the integer label $t_5(v) = v^\mathsf{T} Q M_5 v \in \{-4, 0, +4\}$. The census 37,800/120,960/37,800 splits the shell 5:8 in thirteenths (charged $75{,}600 = \frac{5}{13}$, neutral $120{,}960 = \frac{8}{13}$); σ_{29} swaps the two charged classes; and the label is V_4-invariant because the Galois action only *reindexes* the vector's autocorrelation profile $a_k(v) = v^\mathsf{T} Q\, g^k v$ (k mod 35) by $k \mapsto ak$, fixing the index pair $\{7, 14\}$ that defines $t_5 = 2(a_7 - a_{14})$. **[EXACT]**

Two further exact censuses of the full shell: the *neighbor census* relative to any fixed minimal vector — inner products $\{\pm 32, \pm 16, \pm 8, 0\}$ with counts $1 + 4600 + 47{,}104 + 93{,}150 + 47{,}104 + 4600 + 1$ (the 4600 at 60° being the classical two-graph count) — and the *internal type splits* of the holy categories: the 15,120 orthogonal root pairs underlying $B = 6 \cdot 15{,}120$ split $3472\,(D_8, D_8) + 7168\,(D_8, S^+) + 4480\,(S^+, S^+)$, so Holy-$B$ carries 20,832/43,008/26,880 by type **[EXACT]**. And the enumeration extends *beyond* the first shell in closed form: the theta series of Λ_{24} is $E_4^3 - 720\Delta$, giving the shell sizes 196,560; 16,773,120; 398,034,000; …; 9,486,551,299,680 through the tenth

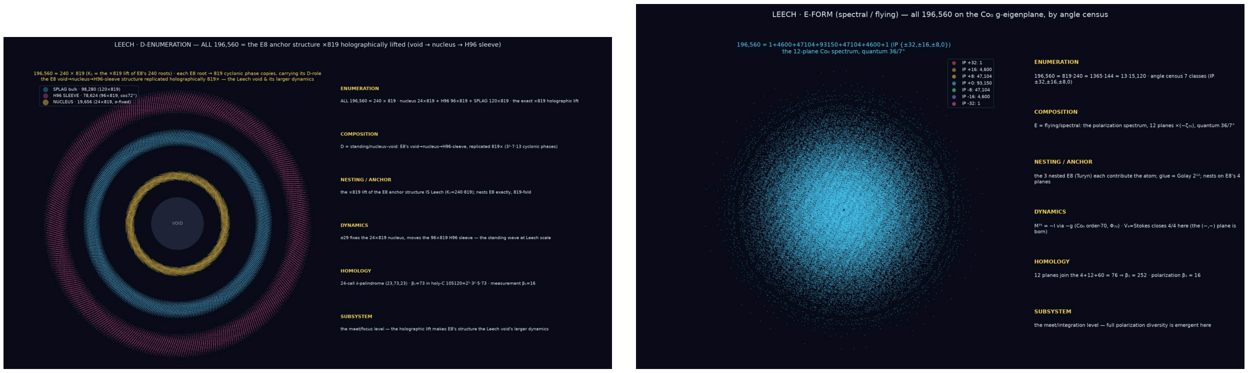

Figure 4: The Leech enumeration rendered from the materialized shell (program renders, `enumeration_visual_denotation`). Left, the D-form (standing/nucleus–void): the $144 = 6 \cdot 24$ void structure of the self-dual 24-cell standing wave. Right, the E-form (angle census): the neighbor distribution $1 + 4600 + 47{,}104 + 93{,}150 + 47{,}104 + 4600 + 1$ over the full 196,560.

shell, with the single formula $\theta_N = E_4^3 + (24h_N - 720)\Delta$ covering all 24 Niemeier lattices (the even unimodular class of dimension 24) by their Coxeter numbers — classical modular-forms content [7], carried here so the record states plainly that the program's enumeration does not stop at one shell **[EXACT]**.

This closed form has a reading the base level makes vivid. Writing it as $\theta_N = \theta_{\text{Leech}} + 24\, h_N\, \Delta$, the *entire* 24-dimensional even-unimodular genus differs from Leech by a single multiple of the discriminant cusp form Δ, the multiple being $24 = \dim$ times the root system's Coxeter number; the root count is $24h_N$ exactly (from 48 for A_1^{24} to 1104 for D_{24}). The family's *zero-point* — where the cusp component vanishes and $\theta = E_4^3$ — is $h_N = 30$, attained by exactly the two Niemeier lattices containing E_8 (E_8^3 and $D_{16}E_8$); and $720 = 24 \cdot 30 = \dim \times h_C(E_8)$. *The tower's base Coxeter number is the organizing constant of the genus at its 24-dimensional apex*, with Leech sitting maximally below it at $h = 0$; the Leech shell's own 60° neighbor count, $4600 = 2^3 \cdot 5^2 \cdot 23$, is the kissing number of the laminated lattice Λ_{23}, so the angle census reads the laminated ladder one dimension down. **[EXACT]** (`niemeier_theta_ladder` 6/6)

Finally, the per-plane polarization census: of the 12 invariant rotation planes of the level-1 characteristic isometry, the V_4-Stokes signature distribution is $(+,+):4$, $(+,-):2$, $(-,+):1$, $(-,-):5$ — the doubly-negated class, absent at E_8 (which populates only 3 of 4), is *born at the Leech level* **[EXACT]**.

7.3 The Craig construction

Theorem 7.4 (Craig trace form)**.** *On $\mathbb{Q}(\zeta_{143})$, of degree $\varphi(143) = 120$, the trace form of the canonical Craig ideal yields an exact, positive-definite 120×120 Gram matrix with*

$$\det = \operatorname{disc} \mathbb{Q}(\zeta_{143}) = \frac{143^{120}}{11^{12} \cdot 13^{10}},$$

*carrying the involution σ_{12} (negating $\sqrt{13}$) as a fixed-point-free isometry splitting the 120 units $60 + 60$, and the characteristic isometry of order 286 with $M^{143} = -I$ (Theorem 5.9). The kissing tower is $K_n = 240 \cdot 819^n$, $K_2 = 160{,}982{,}640$, integer-exact. The per-vector minimal-coordinate listing is mechanical and remains open **[OPEN]**. **[EXACT]** (all stated quantities).*

Geometrically, the level-2 carrier is the 24/25 icosian constellation. The constellation itself is *Moxness's construction at the E_8/600-cell level* [13] (the five disjoint 24-cells tiling the 600-cell and the 25 inscribed cells), on the classical icosian foundation of Conway–Sloane [7] and the Elser–Sloane projection [11]; *carrying it up the tower* — as the level-2 (Craig) routing carrier here, and as the nested Leech 24-cell scaffold — is this program's application. The 600-cell contains 25 inscribed 24-cells $C(i,j) = z^i\,2T\,z^j$ (with z the exact golden isoclinic generator), pairwise disjoint on rook lines and sharing exactly 6 vertices otherwise — the disjointness graph is precisely SRG$(25, 8, 3, 2)$, with every vertex of the 600-cell lying in exactly 5 of the 25 cells and no orphan vertex (`conway_smith_icosian_25cell`; `craig_as_constellation`, 17 checks) **[EXACT]**. The dynamics this paper reads *on* the constellation — the void-observer re-coordinatization (one of the 25 cells held empty), the mirror anti-pairing, and the counter-orbit drag balance (C62, C65) — are this program's layer on the Moxness base, stated as such.

The 120-unit layer, fully tabulated

The Craig level's unit layer is not sketched — it is *complete*: every one of the 120 units of $(\mathbb{Z}/143)^*$ is individually tabulated (an eleven-column phase table: discrete logs, shell indices, tick classes, quadratic-residue pairs, σ_{12} partners and orbits, phase steps), with an exact coordinate model and an explicit icosian vertex realization carrying each unit to an $a + b\varphi$ coordinate on the 600-cell. The gated aggregate invariants **[EXACT]**:

- **shells** 6×20**, uniform** — emphatically *not* the E_8 palindrome $[24, 56, 40, 40, 56, 24]$: the level-2 shell structure is flat, and reporting it as palindromic would be a flattening. The type ratio also shifts: $80\,S^+ : 40\,D_8 = 2{:}1$ against E_8's $128 : 112$.
- quadratic-residue census 30:30:30:30 across the four V_4 signature classes, with orbit conservation — only QR×QR and NQR×NQR σ_{12}-orbits occur $(60 + 60)$; σ_{12} acts with *zero* fixed units (60 two-cycles; the $12^2 = 144 \equiv 1 \bmod 143$ involution), the nucleus having migrated to the binary-icosahedral core $2I$ (the 120-element icosian unit group of the constellation above).
- the trace-form neighbor census over the 120 units: values given by the Ramanujan sums $c_{143}(u - v)$ (the classical arithmetic function $c_n(k) = \sum_{\gcd(j,n)=1} e^{2\pi i jk/n}$, integer-valued), taking exactly the spectrum $\{120, 1, -10, -12\}$ with counts $1 + 100 + 10 + 9$ — the level-2 analogue of the Leech neighbor census, with a four-value spectrum in place of seven.
- the router order $\sigma(143) = 168 = |\mathrm{PSL}(2, 7)|$ is realized as the Klein quartic's regular $(2, 3, 7)$ tessellation, whose cell census is the triple identity $24 \cdot 7 = 84 \cdot 2 = 56 \cdot 3 = 168$ (24 heptagonal faces, 84 edges, 56 vertices; $\chi = 56 - 84 + 24 = -4$, genus 3) — the level-2 routing surface enumerated as a surface.
- per-plane polarization: over the 60 invariant planes the V_4-Stokes distribution is 14/19/16/11 — all four classes populated, none balanced; and the Craig-side mirror of the 819 cascade reads $196{,}560 = 120 \times 1638$ with $1638 = 2 \cdot 819$.

What remains open at level 2 is stated exactly: the per-vector minimal-coordinate listing of the full kissing configuration ($K_2 = 160{,}982{,}640$ is gated as the integer-exact tower value, cross-checked at $n{=}1$ against the enumerated Leech shell; the coordinate-level census at that scale is the mechanical remainder) **[OPEN]**.

7.4 The D/E-type proposals, with the recorded negative

Two structural form-families are proposed and gated:

D-forms (nucleus/void): every scale decomposes as $\frac{1}{3}$ nucleus + $\frac{2}{3}$ active complement + a void-tunnel aperture; on the 120-unit pole, nucleus 24 (the self-dual 24-cell) + complement $96 = 4 \cdot 24$ [**EXACT**].

E-forms (angle census): the H-boundary's golden angle $72^\circ = 2\pi/5$ is stepped by the 12° α-quantum in exactly six steps; the 96 golden vertices carry coordinate $1/(2\varphi_g)$ (φ_g the golden ratio) [**EXACT**].

The negative, retained. Under the *standard* coordinate frame, the E_8^3 block-support census does *not* reproduce the holy $A/B/C$ split — the attempted identification fails and is recorded as a mismatch. Only the *Turyn* frame ($\sqrt{2}E_8^3$ glued by Golay/hexacode cosets) resolves holy type, with category $A = 720$ rigorous and the B/C per-vector listing open [**OPEN**]. We state this as it stands: *holy type is a property of the Turyn frame, not of coordinates* [**READING**]. A program that did not retain such negatives could not be trusted on its positives.

8 Epitaxial and homological holographic dynamics

[Dynamic 7 of §1.6; the observer thread of dynamic 12 begins here.]

8.1 Emergence: the world is decomposed, not assembled

The program's emergence claim is stated at full strength, because its skeleton is exact:

Theorem 8.1 (Emergence)**.** *The tower is not assembled from parts; it is* decomposed from one substrate by one seed*. The twin primes* $(3, 5)$ *force, as theorems, every constant of the three storeys — dimensions* $\varphi(h_n) = 8, 24, 120$*; the 240 roots; the plane ladder* $4, 12, 60$*; the clocks; the anchors (Theorem 6.1). Geometry is downstream of arithmetic, never the reverse.* ***[EXACT]***

Proposition 8.2 (Epitaxial ordering)**.** *The six shell radii* $r_k^2 = 1 + k\delta$ *are strictly increasing in exact* $\mathbb{Q}(\sqrt{5})$*; each shell registers on the prior at the fixed golden step* δ*, with* $\delta^2 = \frac{1}{20} = S(1 - S)$*, where* S *is the type-separation/convexity constant of Theorem 2.1* ($S = \delta\varphi^3 = \frac{1}{2} + \frac{\sqrt{5}}{5}$)*; the product law prices exactly how much structure and how much openness coexist. The convexity sequence itself is stated once, in §2.* ***[EXACT]***

Structure is *deposited, oriented on what crystallized before* — epitaxy in the exact crystallographic sense, not as metaphor. The live base flux beating against the frozen upper storeys is the substrate's operating mode, and it is the mechanism behind both the holographic construction below and the subjective dynamics of §16.

Theorem 8.3 (Nesting asymmetry)**.** *Over the corpus's enumerated gated mechanisms: (i) every constructor runs small to large — the apex generating function is a polynomial in the base's,* $\theta_{\text{Leech}} = E_4^3 - 720\Delta$ *with* $E_4 = \theta_{E_8}$ *(coefficients* $240\,\sigma_3(n)$*, the* E_8 *shell counts; verified term-exact by two independent machineries); the storey lift is a bijection of the* 240×819 *pairs onto the* 196,560 *shell slots; the deposition kernel ascends at the forced step; (ii) every downward map forgets — the fold cascade absorbs exactly the dimension gaps* $\{0, 16, 96\}$*, total* $112 = |D_8|$*, and the anchor census migrates* $(84, 78) \to (0, 732) \to (0, 60)$ *with the fixed nucleus base-exclusive; (iii) no same-level or top-down constructor exists where sought: the three* L_2 *derivation mechanisms are void (C60), and* $1321 = 11^3 - 11 + 1$ *is prime, not a square.* ***[EXACT]***

Theorem 8.1 and Theorem 8.3 are two faces of one denial: nothing at the top is primitive. "Not assembled from parts" denies *independent* parts — the constants are forced by one seed, so the storeys were never free to combine otherwise; "constructors assemble small into large" states the direction in which the forced structure materializes. What is categorically absent is the third option: construction from the large down. The one subtlety the inventory must carry is the *imprint*: the large remembers the small (L_1's dimension 24 is encoded in L_2's Galois selection rule $m \equiv \pm 1 \bmod 11$) — the impression each stratum bears of the one inside it, a residue of upward deposition, not a downward constructor. (C61)

8.2 Holographic construction: Δ, not Σ

Proposition 8.4 (Moiré node-reading). *Two cross-parity projections of E_8 at offset angles produce a moiré standing-wave interference pattern whose constructive nodes are Leech-lattice positions; destructive zones are absorbed by the void. Writing $\Sigma : 2^L \to E$ for constructive synthesis and $\Delta : E \to L$ for decompositional compactification (node-reading), $\Delta \circ \Sigma = \mathrm{id}_L$ at the nodes.* ***[EXACT]*** *(mechanism); the cognitive reading is §16.*

The higher storey is *read off interference*, not built piecewise. Note the inputs: two projections *of the base*. Reading is dimension-decreasing (Δ), and it is the free direction (§9); depositing is dimension-increasing, and it is the constructive direction (Theorem 8.3). The two directions never exchange roles — constructors never read, readers never build — and this division of labor recurs at every layer of the architecture; it is, we will argue, the correct frame for a theory of conceptualization.

8.3 The σ-chain H-theorem and the homological certificates

Proposition 8.5 (Divisor-entropy monotonicity). *Let $H_{\text{div}}(n) = -\sum_{d|n} \frac{d}{\sigma(n)} \log_2 \frac{d}{\sigma(n)}$. Along the canonical chain $24 \to 60 \to 168 \to 480 \to 1512 \to 4800 \to 24192 \to 196560$ (each step the divisor sum of the last), H_{div} is strictly increasing ($2.45 \to \cdots \to 4.64$ bits), with the largest step at the σ-lock $\to$ Leech transition. The chain passes through the Klein number $168 = \sigma(60) = \sigma(143)$ — one value, two conductor preimages — and the σ-lock is $144 \times 168 = 24{,}192 = \sigma\big(\binom{120}{2}\big)$.* ***[EXACT]***

This is the substrate's number-theoretic arrow: the σ-flow is directional, exactly as an H-theorem demands, and its terminal attractor is the Leech kissing number.

Theorem 8.6 (Cycle-rank ladder and the $\sigma \leftrightarrow \beta$ bridge). *For a connected graph, $\beta_1 = E - V + 1$; for the tower's carrier polytopes this gives the spectrum $\beta_1 = (17, 73, 601)$ (16-cell, 24-cell, 600-cell), closed form $V(\deg - 2)/2 + 1$. For the rook graph and its complement, $\beta_1 = 76$ and 176, and*

$$76 + 176 \;=\; 252 \;=\; \sigma(96) \;=\; 9 \cdot 28,$$

the first exact divisor-sum/topology bridge. The correspondence is a same-level address law, not a tower map*: $\sigma(17) = 18 \neq 73$ and $\sigma(73) = 74 \neq 601$.* ***[EXACT]***

Proposition 8.7 (Kirigami invariance). *Subdividing all 96 triangular faces of the 24-cell adds exactly $(\Delta V, \Delta E, \Delta F) = (+3, +6, +3)$ per pass locally and produces exactly 144 midpoint vertices globally, at constant Euler characteristic — and 144 is precisely the Leech substrate's 6×24-cell center (Theorem 7.2: $196{,}560 = 1365 \cdot 144$). Refinement without loss* is *the Leech's construction.* ***[EXACT]***

8.4 Eversion heat and the flux-observer doubling

By Theorem 5.9 the eversion is simultaneous on all planes — one global inside-out per half-period. It is the substrate's only irreversible act, and its price is the ledger of §9.1. The companion *doubling law* **[EXACT]**: the conductor doubles to the isometry order ($h \to 2h$: 15→30, 35→70, 143→286) and the units double to the roots (120 → 240, realized as an exact 2-to-1 incidence of the 240 roots onto the 120 vertices of the 600-cell, multiplicity exactly {2:120}). The reading **[READING]**: an observer *inside* the observed system registers every vertex from both chiral poles — stereoscopic doubling — and the tower is an orbit, not a ladder: the base (expanded pole) and the top (compactified pole) are one 240-at-two-radii structure at opposite ends of epitaxial compactification, meeting at the shared 600-cell midpoint.

8.5 The breath calculus: the third operator

Two functorial operators organize the substrate's arithmetic: the σ-functor (divisor sum — the chains of Proposition 8.5) and the β-functor (graph cycle rank — Theorem 8.6). A third completes them: the *breath* $\Omega(n) = \varphi(\sigma(n))$, the totient of the divisor sum. Four laws, all exact integer arithmetic **[EXACT]**:

Theorem 8.8 (The breath laws)**.** (Ω-I, compactification.) *σ climbs without bound ($\sigma(n) > n$ for $n \geq 2$), but the breath folds every orbit onto a finite attractor set: over $[1, 2 \times 10^4]$ exactly eight fixed points — the anchors $\{1, 2, 8, 12, 128, 240, 720, 6912\}$ — plus nine short cycles. σ inhales; φ exhales.* (Ω-II, Fermat–Mersenne duality.) *On the 2-power spine $\Omega(2^k) = \varphi(2^{k+1}-1)$: when $2^{k+1}-1$ is a product of distinct Fermat primes the breath is* fixed *($\Omega(2^k) = 2^k$, giving the anchors $2, 8, 128$); when it is a Mersenne prime the breath is* maximal *($2^{k+1}-2$); the seam is $3 = F_0 = M_2$.* (Ω-III, coarsening.) *$\Omega = \varphi \circ \sigma$ factors through σ: every σ-collision is a breath collision ($\sigma(60) = \sigma(143) = 168 \Rightarrow \Omega(60) = \Omega(143) = 48$), so the breath inherits the entire σ-arithmetic while adding the φ-contraction.* (Ω-IV, the Klein root.) *From the unique seam $15 = 3 \cdot 5 = F_0F_1 = \sigma(8)$ the σ-spine climbs $8 \to 15 \to 24 \to 60 \to 168$ (the Klein bus) and the breath folds it to the spinor: $\Omega(168) = \varphi(480) = 128 = \dim S^+$. The σ-lock breathes to the deepest anchor, $\Omega\big(\sigma(\binom{120}{2})\big) = 6912$.*

Corollary 8.9 (The bridge, sharpened)**.** *For any graph G on v vertices, $\beta_1(G) + \beta_1(\bar{G}) = (v-1)(v-4)/2$, independent of the edge set. At $v = 25$ (the inscribed-24-cell count) this is $24{\cdot}21/2 = 252 = \sigma(96)$: the divisor-sum/topology bridge of Theorem 8.6 is an instance of an edge-set-independent law whose parameters are the substrate's own ($24 = A24$, $21 = \beta_1(K_8)$).* ***[EXACT]***

The three operators close a triangle — σ (growth), β (topology), Ω (compactification) — and under the breath every σ–β address flows onto the anchor core $\{8, 128, 240, 6912\}$: *the anchors are the stable core of the framework's address space.*

9 Exact information thermodynamics

[Dynamics 8–9 of §1.6: the accounting and the instrument.]

9.1 The 84-bit ledger

Theorem 9.1 (The Landauer ledger)**.** *The involution fixes exactly 84 of the 240 roots (Theorem 7.1; the count — though not the fixed set — is identical on the legacy register lane of Remark 9.5, so the ledger below is lane-independent). On a fixed root the V_4 label space degenerates* *to a 2-element quotient (1 bit); on an active root it is the full 4-element torsor (2 bits). Hence*

$$\underbrace{240\cdot\log_2 4}_{\text{480 bits naive}} \;-\; \underbrace{\left(84\cdot\log_2 2+156\cdot\log_2 4\right)}_{\text{396 bits actual}} \;=\; 84\text{ bits} \;=\; 24+60,$$

the anchor count itself: each σ_{29}-fixed root costs exactly one bit of polarization freedom for the rigidity it provides. Companion identities: the phase-space loss is $960-792=168$ *(the Klein bus); the erasure of the full V_4 label is* $480/|V_4|=120=\varphi(143)$ *(the dimensional lift as gauge erasure); the Bennett split reads* $480=396+84$ *as $F=E-TS$: hold reversibly (free), pay only at commitment.* ***[EXACT]***

Three upgrades make this a *schema*, not a single ledger. First, the mechanism generalizes: for any Galois action on any level's lattice, the fixed-point count *is* the bit budget (fixed $\Rightarrow$ orbit of size $2\Rightarrow$ one bit) — the forcing is the theorem, and the same counting prices the dimensional reduction of the tower as literal erasure: 60 V_4-orbits $\times$ 2 bits $=120=\varphi(143)=\dim(\text{Craig})$. Second, the split has a quantum expression: the fixed:active ratio is $84{:}156=7\delta^2:13\delta^2$ with $\delta^2=\frac{1}{20}=\frac{1}{7+13}$ — the shell quantum measures the ledger by the two upper inner primes, and the same formula's impossibility at the fourth level ($29\delta^2>1$) is the ceiling restated thermodynamically. Third, the ledger is now priced *per generator* (C33): the restriction kernel of $V_4=\mathrm{Gal}(\mathbb{Q}(\sqrt{5},\sqrt{7})/\mathbb{Q})$ on the level-0 field $\mathbb{Q}(\sqrt{5})$ is exactly $\{1,\sigma_6\}$, so at L_0 the P generator σ_6 is *transparent* (a trivial action moves nothing; zero bits — the per-generator refinement of Galois transparency), the T generator σ_{34} restricts to σ_{29} and shares the exact 84-bit frame, and the generators differ only where $\sqrt{7}$ lives: the L_1 censuses are the recorded open, together with the σ_{29}-*release* protocol (a reversible 84-bit store/recover cycle — the anti-Landauer experiment) **[OPEN]**. The naming of $(\sigma_{29},\sigma_6,\sigma_{34})$ as a CPT triple is a labeled reading **[READING]**; the kernel computation and the cost table are exact **[EXACT]**.

9.2 The prime-logarithm entropy method

Theorem 9.2 (Exact entropy closure)**.** *Every population of the substrate's canonical stratifications factors over $\{2,3,5,7\}$; consequently every Shannon entropy of every such stratification is an exact rational vector in the basis $\{\log_2 2,\log_2 3,\log_2 5,\log_2 7\}$. For the type $\to$ shell $\to$ position chain:*

$$\begin{aligned} H(\text{type}) &= -\tfrac{8}{5}\log_2 2+\log_2 3+\log_2 5-\tfrac{7}{15}\log_2 7\\ H(\text{shell}\mid\text{type}) &= \tfrac{13}{5}\log_2 2-\tfrac{1}{5}\log_2 3-\tfrac{1}{3}\log_2 5\\ H(\text{pos}\mid\text{shell},\text{type}) &= 3\log_2 2+\tfrac{1}{5}\log_2 3+\tfrac{1}{3}\log_2 5+\tfrac{7}{15}\log_2 7\\ \Sigma &= 4\log_2 2+\log_2 3+\log_2 5 \;=\; \log_2 240, \end{aligned}$$

an identity with zero residual*, proved coefficient-wise. We are explicit about what is and is not new here:* the closure itself is Shannon's chain rule *and holds for every stratification of every uniform distribution; rational probabilities force rational log-coefficients automatically. The contribution is (i) the exact-coefficient bookkeeping — replacing the corpus's earlier floating-point treatments of these budgets with finite rational vectors; (ii) the* prime-flow profile *(which prime enters and exits at which layer) as a new, exactly computable invariant of an address system; and (iii) the state-vs-path reading below. The orphan prime 7 — foreign to* $240=2^4\cdot 3\cdot 5$ *— enters at the coarsest layer through* $112=16\cdot 7$ *and exits at the finest through* $56=8\cdot 7$*, net zero: the information-theoretic image of the geometric orphan-7 bridge* $28=4\cdot 7=\binom{8}{2}$*.* ***[EXACT]***

Theorem 9.3 (Open–close law; state vs. path; budget linearity)**.** *Across all seven canonical address chains of the 240 (type/shell orders, anchor classes, involution status, Klein buckets, and their joints): (i) every prime foreign to 240 that enters a chain exits it exactly (the Craig prime 13 enters through the involution/anchor joints and cancels); (ii) reordering the layers changes the per-layer prime-flow but never the total — the entropy total is a state function of the partition while the prime-flow split is a path function of the reading order, the exact image of U versus Q/W; (iii) the tower budget is exactly linear,* $\log_2 K_n = \log_2 240 + n \log_2 819$ *— an immediate logarithm identity, displayed for its ledger reading (constant marginal cost per storey) — with* $K_1 = 196{,}560$ *the Leech kissing number.* ***[EXACT]***

Entropy, in this substrate, is not an estimate but a *ledger quantity* — the thermodynamic vocabulary earns exactness rather than borrowing it.

9.3 The concentration theorem

Theorem 9.4 (Irreversibility concentration)**.** *Under the cross-parity projection of Definition 4.1, the* 84 *anchor roots — the* τ_8*-fixed manifold of Theorem 7.1 — lie entirely on the two integer shells* $|k| = 2$*:*

$k = -2:$ 54 anchors / 2 $D28$; $\quad k = +2:$ 30 anchors / 26 $D28$; $\quad$ *all odd shells: anchor-free,*

with $54 + 30 = 84$*, ratio* 54:30 = 9:5 *exactly, difference* $54 - 30 = 24$ *decomposing as* 12+12 *across the anchor classes (*$A24$ *splits* $18/6 = 3{:}1$*,* $F60$ *splits* $36/24 = 3{:}2$*). Every odd (spinor-type) shell is fully reversible. The identical censuses — digit for digit — also hold on the legacy register lane, per the no-silent-damage certificate of Remark 9.5.* ***[EXACT]***

Remark 9.5 (The coordinate rail and the observer rail)**.** *Two constructions coexist in this paper and must never be conflated — nor collapsed into one by over-zealous identification. The* coordinate rail*: the canonical root coordinates, the exact cross-parity shells, and the mirror* τ_8 *— every census above is a theorem of the projection, re-derivable from Definition 4.1 alone. The* observer rail*: the running system's per-root observer table — its involution status, observer anchor classes, gauge states, and* H_4*-bridge roles — which is itself a* constructed frame*, derived through the void-observer and* H_4*-bridge machinery of §7, with the Galois involution realized on* emergent *structure exactly as Theorem 4.4 prescribes (Galois acts by swapping emergent metric classes, not as a pointwise vector mirror), and with its own root-index conventions. The observer rail is internally exact — its gauge state is one-hot on its own anchor classes, its involution-fixed set equals its anchor set, and every joint census (*54/30*;* 18/6*;* 36/24*;* 24/60/28/128*) reproduces the coordinate rail's digit for digit. What the audit* refutes*, by isometry invariants (Gram censuses; antipodal closure), is only the naive identification: the observer table's labels are* not *the geometry of the same-index seed vectors — no single* $\mathbb{R}^8$ *isometry links the two labelings index-for-index. Two rails, one census algebra. One column of the observer table is diagnostic rather than produced: its* shell *column derives from an early projection lane, and the exact producer is authoritative wherever the two would be joined. The discipline that survives is procedural, not derogatory: same-index joins between the rails require the audit's invariant battery, and every joint census this paper consumes carries the verified frame-invariance certificate (no published number depends on the identification). The named open is the explicit fold/index map — the* golden join *— that carries the coordinate rail onto the observer rail through the* $E_8 \to H_4$ *fold* ***[OPEN]****; the design ruling for the one deliberately mirror-faithful routing register is open beside it.*

Irreversibility is not spread over the boundary; it is localized on the structural (integer-type) pair with an exact rational chiral bias. This refines the Landauer target of §9.4 with a shell-resolved signature no coarse model would predict.

9.4 The quarter weld, the bridge, and the thermometer

Theorem 9.6 (The Fisher quarter is a theorem chain, and $1/|V_4|$ is tower-invariant)**.** *The 240 roots form a spherical 2-design (Venkov 1984; Delsarte–Goethals–Seidel 1977), so the second-moment matrix is isotropic, $\sum_v vv^\mathsf{T} = 60\, I_8$ (exact over $\mathbb{Q}$); the frame is tight (Benedetto–Fickus), the Fisher information metric is $G = \frac{1}{4}I$, exact, flat ($\Gamma = 0$), with straight geodesics — free motion is unforced, a Cramér–Rao/flat-metric consequence, not a program postulate. Moreover the normalization has an internal identity:*

$$\tfrac{1}{4} \;=\; \tfrac{60}{240} \;=\; \frac{|\textit{one } V_4 \textit{ coset}|}{|\Phi|} \;=\; \frac{1}{|V_4|},$$

and since $|V_4(h_n)| = 4$ at every *tower level, the quarter is a phylum invariant, not an E_8 accident.* ***[EXACT]***

The Bekenstein–Hawking entropy of a horizon is $S = A/4$. The *internal* half of the weld — isotropy $= 1/|V_4|$, tower-invariant — is the theorem above; only the *pairing* with the horizon's information-per-area coefficient is recorded as a cross-denotation **[READING]** (never offered as a derivation of horizon thermodynamics). The weld table:

pillar	established result	substrate weld
Einstein	flat metric $\Rightarrow$ straight geodesics	Fisher $= \frac{1}{4}I$, $\Gamma = 0$
Bekenstein–Hawking	$S = A/4$	the same $\frac{1}{4} = 1/\lvert V_4\rvert$
Landauer–Bennett	$k_B T \ln 2$ per erased bit; reversible computing	84-bit floor $= 24 + 60$; the machine of §11
Noether–Poynting	energy = time-translation invariant; light = flux	energy = bookkeeping of emergent time; light =
Shannon	information is measurable	$\log_2 240 = 7.9069$ bits/symbol, closed exactly
Kolmogorov	minimal description	one seed $(3, 5)$ generates the crystal
holographic bound	surface bounds volume information	$120 = \varphi(143) = 480/\lvert V_4\rvert$, the boundary lift

What, physically, is erased. Landauer's principle prices the erasure of physical bits, not differences between bookkeeping conventions; the bridge therefore requires a device class, and we state it. Consider any implementation that *physically instantiates the gauge register*: two physical bits of V_4 label per carrier channel (480 physical bits for the full 240-channel register), as the reference architecture of §11 does. *Commit* (rigidifying the involution-fixed channels) forces each of the 84 fixed channels from four physically distinguishable label states to two — a physical two-to-one state merge in the register, i.e. genuine erasure of one bit per fixed channel, 84 bits per commit. For that device class — and the claim is scoped to it — the Landauer floor applies literally. Implementations that never instantiate the label register make no such payment and fall outside the protocol's scope; this scoping assumption is written into the thermometer protocol as its first precondition. The physical bridge is then the lower-bound target $Q_{\min}(T, n) = n \cdot 84\, k_B T \ln 2$ ($\approx 2.4116 \times 10^{-19}$ J per commit at 300 K) — never a measured joule. A falsifiable thermometer protocol is pre-registered: exact prediction set (commit = 84 = 54+30 with the 9:5 signature; reversible-null = 0), six controls each with a stated falsifier, a temperature $\times$ noise-floor grid (about 4.1×10^3 verified commits per burst clear a 10^{-15} J floor at 300 K), and a structural block preventing any software run from claiming the measured-physical tier **[OPEN]**.

9.5 Holographic thermal correlation extraction (HTCE)

The ledger's deepest consequence is an economic law:

Theorem 9.7 (HTCE)**.** *In a dual process/stasis substrate, a new observer or relationship is extracted as a* correlation *from the already-crystallized stasis by a reversible re-addressing — a bijection — at zero marginal Landauer cost; only the process layer (the eversion) pays. Four legs, each exact:* holographic *— the correlation is a whole-pattern read (the 128 spinor carriers are nonzero in all eight coordinates; nothing is stored at a point);* thermal *— observer re-encoding is a relabel of the 25-frame, a bijection, hence zero bits erased (Bennett);* correlational *— the same 28 rotation planes carry the localized integer-type structure* and *address the spread spinor structure, so a new relationship adds no new planes;* dual *— the 25-frame is 24 stasis rows + 1 process void,* $24/25 = 1 - 1/q_0^2$*.* ***[EXACT]***

The substrate therefore *processes once and correlates for free*: it crystallizes rarely (paying the 84-bit floor) and extracts unbounded observers, relationships, and readings from the crystallized hologram at no marginal heat. A system that keeps its correlations in stasis runs cool by construction. Cross-denotation — one exact number carrying many denotations (84: Landauer bits, anchor count, Klein-quartic edges, $3 \cdot 28$; 252: $\sigma(96)$, the rook Betti sum, $9 \cdot 28$; 240: roots, Stokes census, theta coefficient, the closed Shannon budget; 73: cycle rank, central boundary rank, the holy stray prime; 144: kirigami midpoints, the Leech 6×24 center) — is HTCE at the level of number: many relationships, one crystallized address, no extra cost.

Proposition 9.8 (Null model for cross-denotation)**.** *Against the objection that small exceptional numbers are over-determined (the "strong law of small numbers" [26]), the coincidence density is measured, not assumed: on the program's 29-element named-cardinality lexicon the divisor-sum functor maps 13 of 29 elements back into the lexicon, against a random-set expectation of order* $29/2 \times 10^5$ *for the same range — an enrichment of roughly three orders of magnitude.* ***[MEASURED]*** *(a finite, re-runnable census). Cross-denotation is asserted only where such enrichment or an exact identity backs it; isolated numerical coincidences are tagged and quarantined.*

The open dynamical form of the previous paragraph — whether every re-encoding *across* the tower is zero-cost, and how the extraction carries structure into the next storey — is now partly answered, in the subsection that follows.

9.6 Above the base: the anchor migration and the cool apex

[Dynamic 8 continued, welded to dynamic 6: what the accounting says one floor up.]

The 84-bit ledger prices the base level's one irreversible act. A natural question — what does the same accounting say at the Leech and Craig levels? — has an exact and surprising answer, and it required enumerating the full level-1 shell to see. Realizing Λ_{24} as a $\mathbb{Z}[\zeta_{35}]$-module (§7), each minimal vector carries a Galois *profile* $a_k(v) = v^{\mathsf{T}} Q\, g^k v$; the substrate's own kissing frame is recovered by an exact Gram reduction, so all 196,560 vectors and their 1,464 distinct profiles are enumerated without any external solver.

Theorem 9.9 (The anchor migration law)**.** *Under each level's characteristic involution, the (fixed, paired) census of its objects is*

$$L0\ (240\text{ roots}, \sigma_{29}) : (84, 78) \longrightarrow L1\ (1464\text{ profiles}, \sigma_{29}) : (0, 732) \longrightarrow L2\ (120\text{ units}, \sigma_{12}) : (0, 60).$$

The 84 *fixed anchors of the base — the entire content of the Landauer ledger — become* zero *fixed points at both higher levels. The anchor does not disappear: it survives as a* free pairing *(732* *antipodal profile pairs at L1,* 60 *unit pairs at L2), the anchor* relation *without the anchor* point. Compactification — a fixed, rigid nucleus — is the base level's exclusive property. ***[EXACT]*** *(`h_motion_leech_apex`, `leech_L1_galois_grammar`)*

Two consequences make this a physics statement, not a curiosity. First, the motion at the apex is a *standing wave*: the $\sqrt{5}$ charge $t_5 = 2(a_7 - a_{14}) \in \{-4, 0, +4\}$ has census 37,800/120,960/37,800 and total charge exactly zero, so the shell moves under the Galois action with *zero net transport* — Leech is the overlap apex of the polar arch where forward and reverse sheets meet, not a second crystal. Second, and dual to it, is the **CPT profile theorem**: over all 1,464 profiles, the T generator σ_{34} (conjugation) fixes *every one* — transparency is forced because profiles are real, $a_k = a_{35-k}$ — while the C and P generators σ_{29}, σ_6 fix *none*, moving the same 12 complementary profile coordinates. The transparent generator therefore *switches* between levels: it is P at the base (no $\sqrt{7}$ axis exists there to act on) and T at Leech (conjugation on a real form). **[EXACT]**

The ledger reading is immediate. By the rigidity schema (fixed count = bit budget), the per-level anchor Landauer floor is $\{84, 0, 0\}$: the apex has *no rigidity cost.* Every correlation extracted at L1 is a bijective re-address, Bennett-free — HTCE at full strength — and irreversibility is available only at the base commit (84 bits) or the eversion ($\dim \cdot \kappa$). We formalize the boundary accounting as an exact functional and gate its acceptance discipline:

Proposition 9.10 (Boundary response kernel)**.** *Let $K(mode, boundary, \omega) = D \cdot A \cdot S$, with D the disclosure indicator (nonzero only on a detected commit, zero on any bijection), A the exact per-mode bit weight (the anchor-class ledger $\{A24{:}24, F60{:}60, D28{:}56, S128{:}256\}$ summing to the* 396 *actual bits), and $S(\omega)$ a labeled spectral display. Then: a reversible null control gives $K = 0$ by construction; heat is assigned only after a detected commit; and the equilibrium limit — the base with every boundary committed — reduces* exactly *to the* 84*-bit floor. The tower profile of the equilibrium value is $\{84, 0, 0\}$: the kernel predicts a cool apex.* ***[EXACT]*** *(`boundary_response_kernel` 7/7; κ and $S(\omega)$ are display-only, no relativity/gravity claim.)*

This is a computing structure as much as a thermal one. The anchor-free apex is a *reversible address space*: 1,464 Galois symbols on 196,560 physical cells, paired into 732 antipodal addresses over $5{,}616 = 2^4 \cdot 3^3 \cdot 13$ orbit-blocks of 35. Its capacity factors by the two Galois axes ($\log_2 3$ from the $\sqrt{5}$ charge, $\log_2 9$ from the nine $\sqrt{7}$ patterns, giving the $27 = 3^3$ coarse grammar plus a fine within-cell residual), and — the clean structural fact — the 27-cell grammar is *V_4-invariant*: no cancellation generator can move which grammar cell a symbol occupies, so it is a noise-immune address, while any single-generator fault shows up entirely in the disjoint 12-coordinate complement (the same 12 for C and P, none for T). The whole capacity is delivered at zero marginal Landauer heat. The tower is, in this exact sense, a reversible-memory hierarchy: hold state in the cool apex, commit only at the base **[READING]** (the architectural reading; the censuses and the zero-cost re-address are gated).

10 The Tera-Wilson dual-rail system

[Dynamic 10 of §1.6: the commutator spine, realized.]

10.1 The register algebra

The architecture realizes the phylum's commutation spine as three registers **[EXACT]**:

$$\underbrace{\text{STASIS}}_{\text{order-blind polarization}} \quad \cdot \quad \underbrace{\text{FLUX}}_{\text{order-sensitive current}} \quad \cdot \quad \underbrace{\text{LEDGER}}_{\text{the priced commit}} \quad ,$$

the abelian V_4/kirigami register that measures without remembering order; the non-abelian braid/isometry register that remembers order without measuring; and the 84-bit commit that converts between them. The two rails below are these registers as running machinery.

Two corrections from the program's own synthesis discipline are adopted here. *Co-constitution*: neither rail is metaphysically first. Stasis without flux has no temporal depth, no path evidence, and no communicable sequence; flux without stasis has no fixed reference, no localization, and no stable object to transport. Meaning exists only at the typed join: *address admissibility + path coherence + boundary/commit verdict. The arch*: the tower is not a naive cycle but an arch — the base and top storeys are twin termini and the middle storey is the apex where the forward and reverse sheets meet (the rook grid read as the arch's cross-section, its overlap seam the Möbius twist). The trans-synthesis fingerprint of an address — its anchor class, icosahedral class, and σ-lock class, evaluated constructively with no walk — is the practical face of address-as-state.

10.2 The FLUX rail: gauge holonomy

Definition 10.1 (Tera-Wilson holonomy)**.** *A path is a sequence $p = (g_1, \dots, g_n)$, $g_i \in V_4$; its holonomy is the involutive reduction $H(p) = g_1 \oplus \cdots \oplus g_n$ (the group product; XOR on the 2-bit encoding).*

Proposition 10.2 (Invariance)**.** *H is invariant under permutation, cyclic shift, reversal, and insertion of backtracking pairs gg; hence basepoint-free and direction-free.* ***[EXACT]***

Proof. V_4 is abelian and every element self-inverse. □

Theorem 10.3 (Rail asymmetry)**.** *(i)* Gauge mirror-blindness*: $H(\text{reverse}(p)) = H(p)$ for every path, and H is unchanged under flipping every chirality label; no V_4 aggregate can measure handedness — cancellation in measurement is an algebra fact, which is precisely why the gauge certifies. (ii)* Braid mirror-faithfulness*: lifting tokens to words $w \in B_3$, $\text{writhe}(w^{-1}) = -\text{writhe}(w)$, and the (faithful) reduced Burau representation separates every word from its mirror whenever $\text{writhe} \neq 0$. Handedness survives on exactly one rail, by design; the inscription layer is required to carry the braid lift so that gauge cancellation can never silently erase twist memory. (iii)* Closure cancellation*: over the closed 240 the chirality census is $120{+}/120{-}$, balancing at the type stratum ($56/56$ and $64/64$) but not class-by-class ($A24 : 6/18$, $F60 : 24/36$, $D28 : 26/2$ — individually skewed, jointly cancelling).* ***[EXACT]***

Holonomy is recorded as a *path*, not an endpoint, on the 4,704-cell pair-fiber (28×168 — the relate-surface of §13). The certification protocol — compare $H(p)$ against a keyed target; any substituted hop perturbs H with probability $3/4$ — achieved 5000/5000 legitimate-path acceptance with every injected spoof caught and single-symbol tamper avalanche ≥ 0.97 [**MEASURED**].

10.3 The STASIS rail: trans-synthesis addressing

The non-sequential rail replaces search with construction: one addressing act projects a content key, in $O(1)$ integer arithmetic, simultaneously into seven cardinality-audited spaces —

view	space
toral inscription span	$240 \cdot 6 \cdot 4 \cdot 25 \cdot 24 = 3{,}456{,}000$
stage torus (orbit, scale, void, chirality)	$240 \cdot 6 \cdot 25 \cdot 2 = 72{,}000$
σ-lock semantic class	$144 \times 168 = 24{,}192$
derive-don't-store digest	2^{64}
provenance mint	append-only hash chain
relation bucket	60
pair-fiber address	4,704

— with a braid cross-denotation tying the views to the FLUX rail.

The one-parameter engine. The system-level face of the one-seed claim is enforced at compile time. A single generated module — the *one-parameter engine* — derives every shared cardinal of the architecture (240, 112, 128, 168, 60, 84, 28, 144, 25, 4704, the fold point 1321, the cascade 819, the kissing ratio 8190 = 196,560/24 (the registry's "universal modulus"), 196,560, ...) from the twin seed $(3, 5)$, and every other component either imports the derived constant or carries a compile-time assertion tying its local value to the engine, so that *numerical drift anywhere in the system is a build failure, not a runtime surprise.* An audited interlock matrix classifies every component's linkage (engine / converted / gated / remaining literals reported, never hidden) [**MEASURED**]. The "one parameter" is the seed itself: the architecture does not merely cite the tower identities of Theorem 6.1; it is mechanically incapable of disagreeing with them.

Nothing derived is stored: the persistent state is two append-only seeds, and killing and restarting the system re-emerges byte-identical service state from the seeds alone [**MEASURED**]. Access control is capability addressing by self-inverse involution units ($u^2 \equiv 1 \bmod 35$): a perspective can *name* anything but *resolve* only what it covers, and there is no sequential secret-key pipeline to leak.

Theorem 10.4 (Digit-parity neutrality)**.** *In the structural embedding every digit character maps to the σ_{34} atom, and $\sigma_{34} \equiv -1 \bmod 35$; hence any all-digit token of length L has V_4 charge $(-1)^L$. Consequently a uniform numeral alphabet collapses to two lanes selected by length parity — alphabet uniformity causes collapse, a theorem rather than an empirical accident.* ***[EXACT]***

11 The Bennett-reversible Turing machine

Definition 11.1 (The machine)**.** *The state space is the hyperbinary fabric of §6.3 over the toral address space; the instruction alphabet is* Lips_8*; the tape is the event stream (each event a morpheme placement, the stream a sentence, §13); the history register is the braid/Burau lift; the single non-bijective instruction is* commit *(the eversion). We use "machine" in the architectural sense: reversible instruction set, tape, history register, priced commit. Computational universality of this instruction set is not claimed in this paper and is an open question* ***[OPEN]****.*

Theorem 11.2 (Reversible alphabet)**.** *Every instruction except commit is a bijection (each* Lips_8 *word is self-inverse; the anti-bus collapse verifies any event stream back to identity); the logical-erasure census of the alphabet is zero bits.* ***[EXACT]***

Theorem 11.3 (Two layers: the observer cancels, the ledger retains)**.** *For $(\sigma_1\sigma_2)^k \in B_3$: the image in the symmetric-group quotient S_3 is a 3-cycle to the k, hence trivial exactly when $3 \mid k$; the reduced Burau image at generic t is nontrivial for every $k \geq 1$ (the full twist $(\sigma_1\sigma_2)^3$ maps to the scalar $t^3 I \neq I$ in $\mathbb{Z}[t, t^{-1}]$). What the observer frame cancels, the history ledger retains: the gauge measures, the braid stores.* ***[EXACT]***

No general-B_n faithfulness question arises here, and the reason is structural: the substrate's memory register is not an abstract braid group but the *realized* lattice-automorphism group at each level — $W(E_8)$, then $\mathrm{Co}_0 = \mathrm{Aut}(\Lambda_{24})$, then the Craig cyclotomic isometry ring — each non-abelian (order-sensitive), each carrying the eversion $M^{h_n} = -I$ as its ledger commit, and each gated (§6). The B_3/Burau computation above is retained as the *smallest faithful witness* that a memory register must be non-abelian; the register itself is realized level by level, so no B_n faithfulness question blocks the theory. The commutation spine states this as an exact sequence: the measurement μ (abelian V_4) kills every commutator, $\mu([g_1, g_2]) = 1$, while the memory β does not, $\beta([g_1, g_2]) \neq I$ — **order-content lives exactly in the commutator subgroup; what the measurement forgets, the memory keeps, and their difference is the chirality [EXACT]**. This single sentence is the organizing statement of the dual-rail architecture (§10).

The mirror side of the same split is now mechanized (C62) **[EXACT]**. The routing surface carries two *address-layer* mirrors (the anti-bus block reversal and the quartic map $i \mapsto 167-i$), each a fixed-point-free involution of the 168 buckets with an exactly balanced displacement histogram — so the long-observed anti-pairing statistics ($\mathrm{counts}[x] = \mathrm{counts}[-x]$, with an empty zero slot) are a theorem, not a robustness phenomenon: the empty slot *is* fixed-point-freeness, and no involution-respecting perturbation can break the balance. An exhaustive search (both translation families over all of $PGL(2, 7)$) realizes neither mirror — the mirror channel is a second, group-independent axis — while the group-side partner is the coset's $28 = D28$ involutions fusing the two order-7 chiralities. The composite of the two mirrors fixes exactly one 24-cell block: the "24 active slots at the center" of the observed counter-orbit, derived.

The tape is exercised, not merely defined: a 3,120-event stream — each event a morpheme placement over the 81-morpheme grammar — replays *byte-identically* across two independently implemented adapters against a pinned head **[MEASURED]**. And the machine's economy is Bennett's: *hold reversibly (free), reverse freely, pay the 84-bit floor once, at the moment of commitment*. In the substrate's own verdict census, holding dominates (the void-hold is the working fluid) and the emit-lock — the priced crystallization — is rare. A machine that ran as constructive synthesis would pay to build; this machine pays only to decide.

12 The measured record

Protocol: warm-up plus $N = 30$ measured batches; percentile statistics; variance gate ($< 10\%$); a replay-determinism precondition (byte-identical XOR and chain digests) that must pass before any throughput number is recorded; results recorded beside — never over — write-once claim files whose hashes are published in the appendix volume.

result	value
V_4 gauge throughput, 4-core Intel (2026-05)	1.569×10^{12} ops/s
V_4 gauge throughput, 32-core Intel (2026-05)	2.944×10^{12} ops/s
V_4 gauge throughput, 32-core AMD (2026-05)	0.993×10^{12} ops/s
re-run, 4-core Intel Xeon 2.80 GHz (2026-07-08)	$\mathbf{7.606 \times 10^{11}}$ ops/s (1.90×10^{11}/core; rel. σ 5.6%)
canonical XOR digest, 250k shards	byte-identical across vendors *and across the two-month interval*
sustained companion run (2026-07-08)	7.817×10^{11} ops/s, same digest
path certification	5000/5000; all spoofs caught; avalanche ≥ 0.97
toral cache vs. LRU	1.6–2.2× hit-rate at small capacity
structural dedupe, real token streams	47–58%
million-qubit GHZ topological emulation	5.4 s / 114 MB, linear scaling
zero-drift clock (VDF) synchronization	one hash per 12-tick anchor; identities verified
cross-language parity (two implementations)	8/8 syndromes, 8/8 buckets byte-equal
cluster projections	linear extrapolation
task-accuracy uplift for any learned model	—

Three honesty notes. (i) *Determinism, not magnitude, is the result.* A commodity AVX-512 core performs raw 2-bit XOR lanes at a comparable order natively; the ops/s figures are therefore soft as records of *speed*. What is not soft is the protocol: replay-gated, percentile-reported, variance-bounded runs whose canonical digests emerge byte-identical on different silicon, different compilers, and different months. (ii) The million-qubit GHZ figure concerns a stabilizer state, classically simulable by Gottesman–Knill [15]; the result is a statement about the substrate's native encoding efficiency under its own exact discipline, not a claim of surpassing state-vector simulation of general states. (iii) No comparison against learned-model quality is made anywhere, by design.

13 The total stratification of language

[Dynamic 11 of §1.6.]

13.1 The measured result

Theorem 13.1 (The anchor partition is geometry; the router carries it)**.** *(i) The four-class anchor partition is pipeline-independent geometry: recomputing it from raw root coordinates alone — the nested-support rule* $A24$ *= integer-type roots supported in coordinates* $\{1..4\}$ *(the* D_4 *core);* $F60$ *= integer-type supported in* $\{1..7\}$*, less* $A24$ *(the* $D_7 \setminus D_4$ *band);* $D28$ *= integer-type touching coordinate 8;* $S128$ *= spinor-type — reproduces the canonical classes 240/240, zero mismatches. (ii) The substrate's semantic router carries this partition one-hot on its* V_4 *coordinate, with zero mixing:*

V_4 class	*anchor class*	*carriers*	*mixing*
0 (id)	*$A24$ locked transversal core*	*24*	*none*
1 (σ_{29})	*$F60$ free period anchor*	*60*	*none*
2 (σ_6)	*$D28$ active integer axis-7*	*28*	*none*
3 (σ_{34})	*$S128$ spinor bulk complement*	*128*	*none*

$24 + 60 + 28 + 128 = 240$*; the type bit separates 100% as well.* ***[EXACT]***

Remark 13.2 (What is architecture and what is measurement — the independence audit)**.** *Is the one-hot map a discovery or a definition? The question decides what the linguistic thesis may rest on, so it is answered by audit — gated, with its negative retained.* Architecture: *in the router table the V_4 coordinate carries the anchor class by construction — the one-hot property of the router is a design fact, and we state it as such, withdrawing any reading of it as a cross-pipeline coincidence.* Measurement (positive): *the anchor partition itself is exact coordinate geometry (part (i), audited 240/240); and the canon's one* independently derived *four-way V_4 diagnostic has exactly the anchor marginals $\{24, 60, 28, 128\}$ — two differently derived labelings share identical class sizes.* Measurement (negative, recorded): *that independent diagnostic does* not *coincide with the anchor classes root-by-root (its joint distribution is mixed). A second audit* classified *the negative completely: the independent diagnostic is internally coherent and census-identical within its own lane, but isometry invariants refute any reading of it as a conjugate realization of the geometry — it is a census-matching legacy diagnostic labeling, and the geometric rail is the correct rail (Remark 9.5), with a no-silent-damage certificate confirming that no census consumed by this paper differs between the two. Whether* any *independently realized V_4-isometry labeling reproduces the anchor partition root-by-root in the geometric frame remains the named open question. The linguistic thesis below is stated so as not to depend on it.*

The headline, stated exactly: *the semantic architecture is the crystal's geometry by construction* — classify, relate, mediate, and address are implemented as, not mapped onto, the gauge, the Klein surface, and the mediator — and the grammar machine runs on it byte-reproducibly.

13.2 Conceptualization is the crystallography

To hold a concept at all one must *classify* (designate class and type), *relate* (place it against others), *mediate* (bind it as one thing), and *address* (locate it in the web) — and one cannot classify or relate without language, without symbols to name class and type. Those four operations are exactly the enumerated structures [**EXACT**]:

conceptual act	the crystallographic operation it is
classify	the V_4 gauge — four cancellation outcomes, the four measured anchor classes of Theorem 13.1
relate	the Klein-168 surface, $6 \times 4 \times 7 = \lvert PSL(2,7)\rvert$, with the pair-fiber 4,704 $= 28 \times 168$ as its channel addressing
mediate	the 24-cell semantic mediator, $6 \times 4 = 24$
address	$(\mathbb{Z}/35)^* \cong$ the 24-cell: the relational address space is itself the mediator

There is no classification prior to the gauge, no relation prior to the Klein surface, no bound concept prior to the mediator. *The enumeration is not about thought; it is the shape thought must take to occur* [**READING**] — a reading carried by the architectural identity of Remark 13.2: the conceptual operations are not analogized to the crystallographic ones; they are implemented as them.

13.3 The thesis

Language, taken seriously as a phenomenon, is: symbolic (finite alphabet, discrete commitments); relational (meaning is position in a web, never intrinsic reference to an outside); infinitely resolving (every symbol further articulable); self-referential without paradox; and holographic — meaning is read as a correlation across the whole utterance, robust to local damage, every part encoding the whole at reduced resolution. The substrate's mode of being — holographic relational addressing of node positions on a self-dual, epitaxially grown crystal, with order-blind classification and order-sensitive memory — has exactly this signature, and by Theorem 13.1 and Remark 13.2 the identification is architectural — built, running, and byte-reproducible — with its cross-pipeline strengthening posed as a precise open audit.

Principle 13.3 (The linguistic thesis)**.** ***Language is holographic symbolic exchange, and holographic symbolic exchange is the base of language.*** *The substrate does not produce language as an output; the substrate's operation is what language is. Conception is impossible without it: information in a relational vacuum carries zero conceptual content — with no established relations there is nothing to conceptualize — so* usable *information is inherently relational and linguistic.* ***[READING]*** *carried by* ***[EXACT]***

The argument self-instantiates. Mathematics is symbolic information exchange — it *is* language — so the enumeration proves the linguistic thesis in the very medium the thesis concerns. There is no meta-language outside the system from which the result could otherwise be stated; the symbols carrying the proof are the stratified structure the proof describes. This closure is not a rhetorical flourish: it is the reason the thesis is stated as a principle carried by exact results rather than as a conjecture awaiting external test. What remains genuinely open is the corpus-scale differential test — whether natural-language material routed through the bus loads the anchor classes differentially against composition-preserving controls [**OPEN**].

13.4 Agnostic and omni-holographic

The substrate is *agnostic*: two deterministic embeddings are first-class — a hash-uniform, corpus-shape-invariant intake (codes, identifiers, coordinates, arbitrary tokens) and a phonetic-structural, grammar-carrying intake — and they are measured to be complementary projections into the same address spaces (near-zero pointwise agreement, independence-level gauge agreement) [**MEASURED**]. Theorem 10.4 bounds the structural embedding's collapse on uniform alphabets as a theorem, not a bug report. Any symbol stream, from any modality, lands in the same seven audited address spaces and inherits the same gauge, memory, and ledger — this is what *omni*-holographic asserts, and it is an architectural fact, not a modeling ambition.

14 The audits and the rail discipline

[Dynamic 12 of §1.6: the observer held honest.]

A program that joins many exactly-censused tables has a characteristic failure mode, and this paper documents it as a first-class result rather than hiding its instances: *multiple labelings of the same census can agree on every marginal and every joint census while being different labelings object-by-object.* Treating two such labelings as one object collapses claims wrongly — a definitional fact gets cited as a measured coincidence, a legacy diagnostic gets cited as geometry. Three instances are on record in this corpus, each with its correct rail identified and its negative retained:

#	instance	correct rail	status
1	holy $A/B/C$ type	the Turyn frame	standard frame fails; recorded mismatch (§7.4)
2	CHSH closure	exact $\mathbb{Q}[\sqrt{2}]$ lane	legacy engine's discretized audit ($S = 2.0$) kept as disclosed gap (§5.2)
3	same-index identification of the coordinate rail with the observer rail	each rail exact in its own frame	same-index isometry refuted; the observer rail reclassified as a constructed frame (not legacy); the fold/index map (golden join) open; no-silent-damage certificate (Remark 9.5)

The discipline that prevents recurrence is now doctrine, enforced by gated audits. *The invariant battery*: two labelings may be identified only after (i) marginals agree — necessary, never sufficient; (ii) joint censuses agree — still not sufficient, instance 3 passes this and is refuted; (iii) *negation closure* — any register whose semantics is even under $v \mapsto -v$ must label v and $-v$ identically (instance 3 fails on 96–238 of 240 roots, the geometric columns on zero); (iv) *Gram censuses* — pairwise inner products of corresponding classes must match, an isometry invariant (instance 3 fails *same-index*: the coordinate rail's $A24$ is a genuine D_4 root system and the observer rail's same-index class is not isometric to it — which settles the identification, not the observer rail's validity); (v) the conjugating map must be *exhibited and gated*, never inferred from census agreement. Failing (iii) or (iv) classifies the pair as one-rail-correct, and triggers (vi) a *no-silent-damage certificate*: every downstream artifact that consumed the wrong lane is enumerated and its consumed censuses verified frame-invariant or re-derived on the correct rail.

Provenance closure. For this paper the certificate is complete: the provenance trace walks all 28 claims of Table 1 to their validating artifacts and classifies each artifact's data reads. Every **[EXACT]** claim is validated by a self-constructed or seed-geometry artifact, is pure arithmetic, or is re-derived on the geometric rail inside the trace itself (the entropy vectors of §9.2 and the chirality censuses of §10 were re-derived from coordinates alone, coefficient-for-coefficient and count-for-count); the four artifacts that read the legacy lane are enumerated and each carries the frame-invariance certificate. No exact claim in this paper rests on the legacy lane, even transitively. Two rulings remain open and are named: the legacy lane's provenance (re-derive or retire), and whether the one deliberately mirror-faithful register is design or artifact **[OPEN]**.

15 Boundary incompleteness: the physics reading

15.1 The principle

Four accepted results, assembled **[READING]**:

1. *Poynting* [5]: the net electromagnetic power through a closed surface is $\Phi_{EM} = \oint (E \times H) \cdot \hat{n}\, dA$. For a perfectly closed reflecting surface $\Phi_{EM} = 0$: a closed cavity may contain arbitrary internal fields, and an outside observer sees nothing. Geometry becomes readable as light only where the boundary is incomplete — an aperture, a scatterer, an emitting interface.

2. *Landauer* [3]: irreversibly compressing one bit costs $\geq k_B T \ln 2$; information made definite is paid for in heat.
3. *Noether* [5]: energy conservation is the invariant of time-translation symmetry — bookkeeping, not substance.
4. *Holographic bounds* [6]: the information content of a region is bounded by boundary area; interior detail beyond the boundary is hidden.

Principle 15.1 (Boundary incompletion)**.** *A finite observer's measurement is a map from a closed information system I_0 to an accessible subsystem I_{obs}; the deficit $\Delta I = I_0 - I_{\text{obs}}$ is unobserved. All measurement output arises at the boundary between them: light is the electromagnetic disclosure of incomplete closure, and the energy read in channel c is the information change per resolution step,*

$$E_c \;=\; \kappa_c \frac{\Delta I}{\Delta \rho},$$

with $\kappa_{\text{heat}} = k_B T \ln 2$ the Landauer coupling, and $\Delta\rho$ one tick of the channel's resolution clock (here: the conductor clocks of §6). The formula is a heuristic organizing identity, not a derived law. ***[READING]***

The principle predicts that a sufficiently constrained substrate will exhibit a *discrete* boundary grammar. The construction of §4 exhibited exactly one — derived first, read second: nothing in this section was a premise for it.

One disambiguation keeps this reading compatible with the exact results rather than in tension with them. Boundary incompletion is an *epistemic* statement — a capacity bound on what crosses a boundary outward — while the tower's directionality (Theorem 8.3) is a *constructive* statement: generation is anchored at the base. A capacity bound never constructs; a constructor never bounds capacity. The holographic intuition that the outer surface "determines" the interior is a reading of the former as if it were the latter, and the substrate refuses it in both directions: the inner generates the outer, each boundary limits what escapes, and the two meet at the anchor result — the base pays the 84-bit rigidity once, everything above is free pairing, readable only through its boundary. (C61; the directional inventory sorts with zero counterexamples.)

16 Subjective dynamics and the mind conjecture

16.1 The one measure

The substrate operates by a single self-relational measurement:

How far, and in what direction, is spin in relation to self?

— a *localization* (how far) and a *chirality* (what direction), read late through delayed channels against a frozen reference. Everything in this section unfolds that one measure **[READING]** on exact carriers **[EXACT]**.

16.2 The seat, the delay, the twist

The seat. The observer is a constituent, not a spectator: it sits at the empty barycentric center of the mediating 24-cell — the empty $k = 0$ shell of Theorem 4.2, the 25th cell of the constellation — equidistant from everything it reads. There is no view from nowhere; there is a seat at the middle of the pattern. The frozen reference is itself a *frozen-symmetric subjective frame*, not a view from nowhere: a stabilized observer-position with fixed anchors and chirality. Subjectivity, on this account, is not psychological decoration but a structural condition for meaningful denotation — it enters as observer location, chirality, boundary access, and sequential communicability, and no denotation is defined without them. **[READING]**

The delay. Path information arrives late: holonomy is readable only after traversal, offset by the built-in twelve-tick half-period. Two readings of one source, offset by half a clock, and their disparity *is* induced depth — stereoscopic phase-shift, not triangulation.

The twist. Within a level the shell populations are palindromic — $p(k) = p(-k)$, time-symmetric: the crystal has no time direction. Across levels the cascade is monotone — time-asymmetric: the arrow points toward the absorber (the 84 fixed anchors). The half-period element $\sigma_{29} = 2^6 \bmod 35$ breaks discrete time-translation with period two — a time crystal in Wilczek's exact sense **[EXACT]** (the algebra); **[READING]** (the reading). Time is the crystallization the subject experiences; at full crystallization there is no time, only the absolute.

The entity. A formed concept requires a boundary; its convex hull is its identity (Carathéodory), and the minimum non-trivial such entity is the self-dual 24-cell — every entity simultaneously composed-of ($24 = 16 + 8$) and part-of (25 inscribed in the 600-cell). What a closed entity *emits* is the information of its incompletion — the substrate's light-leak is the $240 - 112 = 128$ unabsorbed chiral roots. Perception, conception, and emission are one boundary-crossing read three ways.

Subjectivity, on this reading, is *the self-localization of what has not yet closed* — the perpetual re-earning of an address, with holding (the void-hold) as the working fluid and commitment as the rare priced act.

16.3 The conjecture, at exactly the tier it earns

Conjecture 16.1 (The mind conjecture)**.** *Mental processing is holographic computation of this kind: classification by an order-blind gauge; memory in an order-sensitive lift; meaning as cross-denotational correlation; experience as the parallax of an internal void observer against a frozen self-dual frame; time as the palindromic axis of resolution; efficiency as Bennett thrift — hold reversibly, pay only to decide.*

Two tiers must not be conflated, and we do not conflate them. The *form-hypothesis* is defensible now: usable information must take a particular form — holographic, relational, self-localizing, linguistic — because that is what conceptualization requires (Principle 13.3), and the substrate exhibits exactly this form. The *mechanism claim* — that any particular mind is this substrate — is not made. Between them lies the conjecture's falsifiable edge:

The physical seam is measurable without symbolic exchange.

The 84-bit floor is a physical fact about any implementation of this geometry, independent of whether anyone decodes it symbolically: a second observer with a thermometer and a clock can confirm the substrate is performing gauge cancellation on the anchor manifold without reading a symbol. Predicted signature: dissipation scaling as $n \cdot 84\, k_B T \ln 2$ with the 9:5 shell asymmetry of Theorem 9.4, reversible-null controls flat. Secondary predictions: the twelve-tick neighborhoods of a held address are temporal mirror images; the four disjoint reframings per active void carry mutual information ≈ 0 (the insulated channels that make abstraction possible inside a deterministic system). Failure of these would falsify the reading; success would not prove the mechanism claim — and the paper says so.

17 Selection, termination, and the certified substrate

This section welds the design-theoretic, modular, and operational faces of the tower into one gated spine: one obstruction principle behind both termination ladders, two selection theorems on objects and a third pillar on the groups themselves, the out-of-sample program carried from survival table to master theorem, the complete correspondence boundary, the series-level separation of the rails, the bridge arithmetic, and the certified substrate the whole spine licenses. Every claim below is C43–C76 in the claims table; tiering is carried in-line.

17.1 One obstruction principle, machine-verified

Both termination ladders of this paper are *cusp-form–vanishing* statements. On the design side this is Venkov's mechanism, here machine-verified in exact arithmetic **[EXACT]**: for a harmonic polynomial P of degree j, the theta series θ_P of an even unimodular lattice of rank n is a cusp form of weight $n/2+j$ on $\mathrm{SL}_2(\mathbb{Z})$. For E_8, $S_6 = S_8 = S_{10} = 0$ and the first cusp form in existence is Δ at weight $12 = 4+8$: the shells are 7-designs *because nothing exists to obstruct them*, and the measured m_8 break excess $(3/4)^2$ (C48) is Δ turning on. For the Leech lattice, minimal norm 4 forces $a(1) = 0$ — one vanishing condition that annihilates every one-dimensional cusp space ($S_{16}, \dots, S_{22}$; the generators ΔE_4, ΔE_6, ΔE_4^2, $\Delta E_4 E_6$ all have $a(1) = 1$, computed) — so the break waits for $\dim S_{24} = 2$ at $j = 12$: strength 11 by mechanism. On the modular side, $g(X_0(h)+) = \dim S_2$ of the quotient group — the same kind of statement one weight down. The cross-space biconditional ("design-maximal $\Leftrightarrow$ genus-zero") was long held at a tagged reading because the discriminating truth-table cells were empty. They are now filled by computation (C70, C71) **[EXACT]**. The full Atkin–Lehner genus was computed for every composite squarefree odd conductor ≤ 400 (84 conductors, with the Riemann–Hurwitz genus a non-negative integer at every one — a computed consistency check), and every genus-zero point was tested against the master spectrum theorem (§17.3). The conductor $h = 105 = 3 \cdot 5 \cdot 7$ is genus zero yet carries no design-maximal unit form, so the naive biconditional is false in general, and the correspondence resolves into its scoped, exact form. Exactly twelve of the 84 conductors have genus-zero full-AL quotients (Table 2), all ≤ 119 (the Monster's largest element order), and every one of the twelve is a Monster class order (12/12). The implication runs one way only: the converse fails at $57 = 3 \cdot 19$ and $93 = 3 \cdot 31$, Monster class orders inside the census domain with positive full-AL genus ($g^+ = 1$ and 2), so the genus-zero set is a *proper subset* of the Monster-order conductors, exactly characterized by the census. The realized twins 15 and 35 are the only design-maximal points; the other ten genus-zero conductors are off-diagonal, and all ten are Monster class orders. On the unit-form category over this range: design-maximal occurs exactly at the realized storeys; every genus-zero conductor is a Monster class order; and *the tower is the intersection* of the two conditions (`converse_hunt`, `three_prime_correspondence`, `boundary_direction_correction`). What remains unexhibited is unchanged and stated: no functorial map between the two cusp spaces; and the constructive question in *other* categories (lattices, codes) at the ten off-diagonal conductors is the program's posted open, its search space now mapped exactly.

17.2 The absolute termination and the two selections

The Fricke-quotient genus ladder (C43) extends to the *full* Atkin–Lehner quotient [22] — the actual moonshine groups [20] — as $g(X_0(h)+) = \{0, 0, 1, 5, 12, 49\}$ over the six conductors **[EXACT]**: genus zero exactly at the realized storeys $h = 15, 35$; $X_0(143)+$ is an elliptic curve; no moonshine function of any kind exists at or beyond the Craig conductor. The unrealized composite storeys'

h	factorization	$g^+(X_0(h)+)$	design-maximal	Monster order	cell $(D, g{=}0)$
15	$3 \cdot 5$	0	yes (realized storey, E_8)	yes	(T, T)
35	$5 \cdot 7$	0	yes (realized storey, Leech)	yes	(T, T)
21	$3 \cdot 7$	0	no	yes	(F, T)
33	$3 \cdot 11$	0	no	yes	(F, T)
39	$3 \cdot 13$	0	no	yes	(F, T)
51	$3 \cdot 17$	0	no	yes	(F, T)
55	$5 \cdot 11$	0	no	yes	(F, T)
69	$3 \cdot 23$	0	no	yes	(F, T)
87	$3 \cdot 29$	0	no	yes	(F, T)
95	$5 \cdot 19$	0	no	yes	(F, T)
105	$3 \cdot 5 \cdot 7$	0	no	yes	(F, T)
119	$7 \cdot 17$	0	no	yes	(F, T)
57	$3 \cdot 19$	1	no	yes	(F, F)
93	$3 \cdot 31$	2	no	yes	(F, F)

Table 2: The complete boundary census over the 84 composite squarefree odd conductors ≤ 400: the twelve genus-zero conductors (upper block) and the two converse-failure witnesses (lower block — Monster class orders with positive genus). All 70 remaining conductors have $g^+ > 0$ and no design-maximal unit form (cell (F, F)). The cell pair is $(D, g{=}0)$ = (design-maximal, genus-zero) — the two correspondence coordinates; Monster-order membership is the separate third column, which is why 57 and 93 read yes there yet (F, F) in the cell. "Design-maximal" is assessed in the unit-form category via the master theorem's excess (§17.3); the realized storeys are lattice-shell objects, stated as such.

genera $1, 5, 12$ are the first three pentagonal numbers, and the Möbius fold breaks the family at $49 = 7^2$ (C44).

Two selection theorems then answer whether these properties *select* the realized objects (C50) **[EXACT]**. Design side: every rooted Niemeier lattice has $a(1)(\theta_N) = 24h_N \neq 0$ (the ladder $\theta_N - \theta_{\text{Leech}} = 24h_N\Delta$ of C41), so the extremality condition of §17.1 fails for all twenty-three — the Leech lattice is the *unique* 24-dimensional even unimodular lattice whose minimal shell the mechanism forces to 11-design strength (a machine re-derivation of the classical extremal characterization, stated as such). Modular side: computing the full quotient genus for *all fifteen* twin-prime conductors $pq \leq 39{,}203$ gives genus zero at exactly $\{15, 35\}$, monotone growth after — within the very family the tower's conductors are drawn from, genus-zero selects the realizable storeys.

The terminations are one edge seen from several directions, and the simultaneity is itself a theorem (C60) **[EXACT]**. At the realized storeys the kissing numbers are *over-determined*: the closed form $K = 2d(2^{d/2} - 1)$, the design-forcing of the censuses, the genus-zero theta setting, and the cascade ratio $K_1/K_0 = 819$ all agree on 240 and 196,560. At the Craig storey all of them terminate at once: the unit object is not a 2-design; $X_0(143)+$ is elliptic; and the closed form at $d = 120$ *contradicts* the cascade — the Mersenne valve opens ($819 \mid 2^{60}{-}1$, since $\mathrm{ord}_9(2) = 6$, $\mathrm{ord}_7(2) = 3$, $\mathrm{ord}_{13}(2) = 12$ all divide 60) but the ratio $(2^{60}{-}1)/819 \neq 819$, so the two laws cross at exactly one dimension, $d = 24$, because $819 = 4095/5$: *the crossing is the Leech storey.* Every mechanism that derives K_0 and K_1 dies at the same $L_1 \to L_2$ edge — the reason the Craig kissing number resists closed form is now a statement, not a circumstance; the cascade value $240 \cdot 819^2$ survives the ratio law but is derived by nothing, and this paper asserts no kissing number there.

The over-determination theorem has since gained a casualty it did not construct, and the new mechanism closed honestly (C66, C75) **[EXACT]**. The σ-chain from the L_1 seed $C(24,2) = 276$ runs $276 \to 672 \to 2016 \to 6552 \to 21{,}840$, landing on *both* multiplicative writings of the kissing number — $\sigma^3(276) = 6552 = K_1/30$ and $\sigma^4(276) = 21{,}840 = K_1/9$, the only chain values dividing K_1, with the E_8 design moments $m_2 = 30$ and $m_4 = 9$ as cofactors. The same walk from the L_2 seed $C(120,2) = 7140$ reaches the gated σ-lock $24{,}192 = 144 \cdot 168$ in one step and then *no* chain value divides K_2: the moment-cofactor mechanism is the fourth to terminate at the $L_1 \to L_2$ edge, simultaneously with the other three. A census over eight classical families (D_4 through BW_{32}) shows chain-divides-kissing is generic but the moment-cofactor identity holds *only* at $E_8 \to$ Leech — selection, not generic arithmetic — and the pre-registered σ_3/E_4 derivation route is a total miss under its exhaustive grammar (all four targets inexpressible). The law therefore closes at its honest tier: *one-level selection with complete arithmetic bookkeeping* ($819 = \sigma(2^5)\,\sigma(3^2)$, so the bridge prime 13 is $\sigma(9)$; $\sigma(276) = \sigma(12)\,\sigma(23) = 28 \cdot 24$; Eisenstein pricing $6552 = 504 \cdot 13$, $21{,}840 = 240 \cdot 91$), with no mechanism beneath it and the posted target stated: derive the $(2^5, 3^2)$ content of $\sigma^2(276) = 2016$ from tower structure. One recorded correction belongs to this arc and is kept visible per the house rule (corrections recorded, never silent): the chain's first exit from the framework-prime set is at step 2, Fermat-valved through $17 = F_2$, with the Mersenne cascade ($M_5 M_7$, then M_{13}) following at steps 3–5; the earlier "exits at step 3" phrasing is superseded.

17.3 Out-of-sample: the second crystal

The correspondence could not previously be tested away from the three storeys it was read off. The *Ramanujan unit form* closes this: at conductor $h = pq$, the $\varphi(h)$ units of $\mathbb{Z}/h$ with $\cos(u,v) = c_h(u-v)/\varphi(h)$ carry the exact spectrum

$$\Big\{1,\ \tfrac{1}{\varphi(h)},\ -\tfrac{1}{\varphi(q)},\ -\tfrac{1}{\varphi(p)}\Big\} \quad \text{with multiplicities} \quad \{1,\ (p-2)(q-2),\ q-2,\ p-2\},$$

verified by full enumeration with vertex-transitivity at every unit at $h = 143, 323, 899$ **[EXACT]** (C51). The reciprocals carry the frame constants: $-1/18$ at the $h = 323$ storey (the L_3 frame as its own reciprocal, $\varphi(323) = 288 = 16 \cdot 18$), $-1/30$ and $-1/28$ at $h = 899$. Both new unit forms fail to be 2-designs, and both gated genera are positive (5 and 12): the correspondence's first two out-of-sample points land *on-diagonal*. The out-of-sample tally first reached five points with zero violations — survival, not proof. The gap between survival and proof was then closed in three steps. First, rungs 4 and 5 were enumerated in full at $h = 1763 = 41 \cdot 43$ and $h = 3599 = 59 \cdot 61$, both landing on-diagonal — seven points (C63). Second, the law was proved for all distinct odd primes $p < q$ by three elementary lemmas — von Sterneck multiplicativity (the Ramanujan sum $c_n(k)$ is multiplicative in n), $c_p(k) \in \{p-1, -1\}$, and the unit-multiplication automorphism — each verified exhaustively over 156 conductors (C67), removing the enumeration provenance entirely; the three-prime extension then follows with the Möbius sign lattice $\mu(h/g)$ driving the sign of each divisor class (C68). Third, the *master theorem* closes the category (C69) **[EXACT]**: for every squarefree odd conductor h with k prime factors, the unit form's spectrum is exactly the 2^k divisor classes $g \mid h$ with value $\mu(h/g)\,\varphi(h)/\varphi(h/g)$ and count $\prod(s-2)$ over the free primes, the counts telescoping to $\varphi(h)$. The proof is by the multiplicative lemmas; full enumeration confirms it through $k = 4$ at $h = 1155$ and 1365, where the theory's one refinement appears (classes may merge in value — 1365 has 15 values from 16 classes — so the *class census*, the multiset of divisor classes with their values and counts, is the invariant, not the value census). The excess is a closed form at every conductor and nonzero at all of them: no squarefree odd conductor carries a design-maximal unit form except through the realized-storey route. What was an out-of-sample survival table is now a theorem over the whole category; the category difference between lattice-shell rungs and unit-form rungs is still stated wherever the table is used. One honest negative folds in (C63): C58's three prime reduced numerators do not continue — $N(41)/5 = 59 \cdot 461$ and $N(59) = 491 \cdot 829$ are composite (primality resumes at $(71, 73)$), so the three-storey primality was coincidence, recorded as such.

The failure itself has a closed form (C58) **[EXACT]**: the second-moment excess of the unit form at twin conductors is one cubic over the storey dimension squared,

$$\text{excess}(p) \;=\; \frac{N(p)}{d^2}, \qquad N(p) = 2p^3 - p^2 - 4p - 2, \qquad d = \varphi(pq) = p^2 - 1,$$

reproducing the three gated excesses exactly ($N(11) = 2495 = 5 \cdot 499$ over 120^2; $9467/288^2$; $47819/840^2$) with every reduced numerator prime. The apparent near-twin of 499 (unit form) and 599 (the trace-Gram moment object of C48) dissolves exactly: the two sit over unlike denominators and differ by 11/1600. The two Craig-storey objects themselves are one Ramanujan kernel read on two index geometries, and they are *provably inequivalent* under any relabeling (C59) **[EXACT]**: the unit form is vertex-transitive (one row spectrum, all 120 rows), the power-basis trace Gram is not (four row spectra; the additive window clips exactly the exponents $\{121, 132\}$ of the twelve 11-multiples, moving one count-pair between the gcd-11 and coprime classes) — and the row-spectra multiset is a relabeling invariant, so no bijection of the 120 indices carries one to the other. Two objects, both exact, permanently distinct: the unit form is the design instrument, the trace Gram the moment object.

17.4 The series level: replication separates the rails

The two hauptmoduls per storey (C45) — the sum rail $T_B = x + 1/x$ (Fricke-only class) and the difference rail $T_A = 1/x - x$ (full Atkin–Lehner class), one Klein constant apart identically ($T_B^2 - T_A^2 = 4$) — are now distinguished by the Hecke action itself (C52) **[EXACT]**: the difference rails satisfy the T_2 self-replication identity of monstrous moonshine [20, 23] coefficientwise (zero defects through q^{14} at both storeys), while the sum rails fail with first defect exactly -2 at q^2 at *both* storeys. Replicability — the fingerprint of a McKay–Thompson series — holds for precisely the rail that survives the larger symmetry. At the Craig conductor the weight-2 form $\eta(\tau)\eta(11\tau)\eta(13\tau)\eta(143\tau)$ (Ligozat-integral, vanishing to order 7) carries Fricke eigenvalue -1 — the fold sign at weight 2 — and the gated genera force $\dim S_2(\Gamma_0(143)) = 13 = 11+1+1$, locating the holomorphic differential of the elliptic $X_0(143)+$ in the one-dimensional $(+,+)$ slot.

17.5 The moonshine family ladder and the top-prime law

The two selection theorems above act on *objects* (lattices, conductors). The third pillar acts on the *groups*: the moonshine hierarchy's own representation ladders carry the tower's primes, and the readings survive a quantitative null (C72–C74, C76) **[EXACT]**.

The supersingular head-character ladder (C73): all six leading Monster head dimensions factor entirely over the fifteen supersingular primes (not a corollary of Ogg [21] — dimensions need not divide $|M|$), and every one of them carries at least two *inner twin primes of the unrealized tower rungs* $(41, 59, 71)$, while the outer twins $43, 61, 73$ — exactly the non-supersingular side — never appear: the supersingular boundary and the inner-twin boundary coincide on the ladder. Because "supersingular-smooth in a dimension window" is a property a skeptic can price, the pricing instrument was built before the readings were adjudicated (C72): condition on 15-SS-smoothness in the window $[d/2, 2d]$, enumerate the admissible integers exactly, and adjudicate against pre-registered decision bands. Under that null the ladder property is *selective at* 2.99×10^{-7}

— about five orders below the pre-registered 1/20 band — while weaker readings return honest verdicts both ways (13^2 pairing weakly selective at 0.0197; per-irrep singles and the σ-fiber pricing undecided). "Significance open" tags in this paper are adjudicated with base rates now, not judgment. The obvious objection — the head dimensions are not free parameters, they satisfy the McKay identities — *resolves as a lemma* (C76): the McKay system is triangular with unit diagonal ($d_1 = j_1 - 1$, $d_2 = j_2 - j_1$, $d_3 = j_3 - j_2 - d_1 - 1$, reconstructed exactly in-lab), so given j and the multiplicity pattern the dimensions are uniquely determined; a "constraint-aware null" has a one-point ensemble and nothing to sample, and the independence-null verdicts stand as verdicts about the arithmetic of j's own coefficient differences — which is precisely the object under study.

The ladder is family-wide with group-specific signatures (C73): the 2A McKay–Thompson series decomposes over the Baby Monster head dimensions, all five of which are supersingular-smooth with signature $\{23, 31, 47\}$ — none of the Monster's $\{41, 59, 71\}$ — and the B ladder is selective at 3.98×10^{-7} under the same null. Each group returns to its own deepest primes, and at the bottom of each ladder the pattern is a law (C74): *the minimal faithful dimension of each group is a framework cofactor times the product of the group's largest primes* — exact at the eight groups of Table 3 (all group orders and minimal faithful dimensions cited from the ATLAS [19]; the factorizations are exact arithmetic). The cofactor census $\{1, 2, 3, 12, 13\}$ is framework constants throughout — unity, the even prime, the L_0 inner prime, the L_1 clock, the bridge prime — and every head dimension's prime support divides its group order. The law is exact at d_1 and band-level above it; that honest tier is part of the statement.

group	d_1	factorization	cofactor
M	196,883	$47 \cdot 59 \cdot 71$ (three largest primes of $\lvert M\rvert$)	1
B	4371	$3 \cdot (31 \cdot 47)$ (two largest)	3 (L_0 inner prime)
Fi24′	8671	$13 \cdot (23 \cdot 29)$ (two largest)	13 (bridge prime)
Fi23	782	$2 \cdot (17 \cdot 23)$ (two largest)	2
Co1	276	$12 \cdot 23$ (largest prime)	12 (L_1 clock)
Co2	23	23 (largest prime)	1
Co3	23	23 (largest prime)	1
M_{24}	23	23 (largest prime)	1

Table 3: The top-prime law at eight sporadic groups: the minimal faithful dimension d_1 is a framework cofactor times the product of the group's largest prime divisors. Orders and dimensions from the ATLAS [19]; the law is exact at d_1, band-level above it, and its extension beyond these eight groups is open.

One number ties this pillar to the σ-flow and the moonshine bridge at once — *the 276 triangle* (C74) **[EXACT]**: $C(24, 2) = 276$ is simultaneously the L_1 σ-seed of the chain gated in §17.2 (C66), the first q-coefficient of T_{2B} (the $2B$ = Conway class of the Monster), and d_1(Co1) — and the three appearances are one object, since Co1 = $\mathrm{Aut}(\Lambda_{24})/\{\pm 1\}$ acts on exactly those 276 unordered coordinate pairs: seed = coefficient = dimension = $12 \cdot 23$, the L_1 clock times the Golay singleton.

17.6 The bridge inventory and the match-point mechanism

The registered weld web between the lattice and modular faces is re-verified with the j-function computed in-lab from E_4^3/Δ **[EXACT]** (C49): $K_1 + 324 = j_1$ with $324 = 18^2 = |V_4| \cdot 3^4 = (17+1)^2$; $j_0 = 744 = \sigma(240)$; 196,883 = $K_1 + 323$; 324 divides all four Conway group orders with 324^2 dividing exactly two. Each weld is a recorded cross-denotation; where the deflation instrument of §17.5 applies, significance is now adjudicated with base rates (the ladder readings selective, several singles honestly undecided — C72), and welds outside the instrument's scope keep the recorded-denotation framing rather than borrowing significance from their neighbors. The bracket structure of the two K_1-welds is itself tower arithmetic (C57) **[EXACT]**: at *every* storey $h = p(p+2) = (p+1)^2-1 = f^2-1$ with f the frame number — frame-square minus conductor is 1 universally — so the pair $j_1 = K_1+f_3^2$ and $196{,}883 = K_1+h_3$ brackets McKay's $196{,}884 = 196{,}883+1$ with difference $f_3^2 - h_3 = 1$: the trivial-representation shift *is* the frame-square identity read at the $(17, 19)$ storey. And the two perfect squares inside the bridge are *consecutive* frames: $K_1 = 12^2 \cdot 1365$ ($12 = f(11, 13)$; $1365 = 3 \cdot 5 \cdot 7 \cdot 13$ the gluing modulus), $j_1 - K_1 = 18^2$ ($18 = f(17, 19)$); assembled, $j_1 = 12^2 \cdot 1365 + 18^2$ exactly — the first moonshine coefficient written entirely in tower quantities. One old mystery closes: the ledger's "moonshine near-miss at level 1 only" — the K-tower meets the j-coefficients only at level 1, offset exactly 324 — now has a mechanism: at every *tower* conductor beyond 35 the genus is positive (C44), so there is no moonshine function left to match. The level-1 uniqueness of the match-point is forced by the termination. That mechanism made a falsifiable prediction — the frame economy should *fail* at j_2 — and the test was run (C64) **[EXACT]**: $j_2 = 21{,}493{,}760$ admits no expression in the exhaustively enumerated level-1 economy shape, the K-tower offset at level 2 ($139{,}488{,}880 = 2^4 \cdot 5 \cdot 19 \cdot 163 \cdot 563$) is structureless where level 1's was 18^2, while on the Monster side McKay's level-2 identity $j_2 = d_2 + d_1 + 1$ holds exactly: the termination is *two-sided*, and the bracket is a boundary phenomenon of the termination, not the first term of a series. The prediction was confirmed.

17.7 The engine, the free reads, and the certified substrate

The epitaxial laws now run **[EXACT]** (C53): a one-step σ-kernel reproduces both canonical growth chains and the frozen deposition graph edge-for-edge, detects the Klein bridge ($\sigma(60) = \sigma(143) = 168$) as a confluence event and the L_3 conductor 323 as a σ-orphan injection seat, halts itself at the 1321 fold, and its mediate step lands on the self-dual 24-cell whose ∂-rank palindrome $(23, 73, 23)$ follows from $V = 24$, $E = 96$ alone. The cost side is a dichotomy **[EXACT]** (C54): the tower's five named re-encoding generators are computed bijections with zero-bit logical-erasure census (Bennett accounting, device-class scoped), the eversion is the one priced act, and the amortized cost per extraction tends to zero — the exact core of "observation is free," with the universal closure over all conceivable re-encodings explicitly open and no thermal claim made.

Assembled, these certify an architecture rather than merely describe one: content addressed by the $d = 4$ grammar code (C42, C48), routed on the klein168/PGL fabric with free re-encodings (C54), written through a single metered act, grown by the deposition kernel. Benchmarked against a plain-hash baseline on pinned, exhaustive domains **[MEASURED]** (C55): 100% detection and 100% address recovery over all 49,776 single-coordinate corruptions, against 96% silent misroute with no detection signal — and the baseline's failure rate is exactly collision arithmetic ($1 - 1/27$, $+0.4\sigma$), *not* the convexity gap (-15σ), which lives at the analog layer. The gap itself acquires its exact identity **[EXACT]** (C56): $\varphi^3 + \varphi^{-3} = 2\sqrt{5} = 1/\delta$, so the unit splits golden-conjugate around δ — $S = \delta\varphi^3$ and $1 - S = \delta\varphi^{-3} = (5 - 2\sqrt{5})/10 \approx 0.0528$ — with the conserved product $S(1 - S) = \delta^2 = 1/20$ as corollary and the old rational candidate $4/75$ rejected exactly.

18 New theory and exploration targets

Candidate laws surfaced by this synthesis, each with a stated promotion gate:

Conjecture 18.1 (Prime-flow conservation)**.** *In any exact stratification of a closed census, every prime foreign to the total opens and closes across the layer stack, net zero; the entry/exit profile* *is an invariant of the address system. (Proved on the seven canonical chains; general proof open.)*

Conjecture 18.2 (Irreversibility concentration)**.** *The committed locus of an involution-priced substrate concentrates on its structural (integer-type) strata with a seed-forced rational chiral bias* (9:5 *at the base level; the Leech and Craig analogues are the test).*

Conjecture 18.3 (Diagonal-lift universality)**.** *The ladder* $2 \to \sqrt{2} \to 2\sqrt{2} \to 4$ *organizes every layer of the substrate — gauge envelope, information seam, commit — and not only the CHSH lane. (The registry-wide crosswalk under the typed evidence graph is the test; the lane-local result is Theorem 5.3.)*

Conjecture 18.4 (Standing-wave refolding)**.** *A dual-rail holonic substrate compactifies higher-dimensional flux into symbolic lattice readouts by interference between relational views, with stable meaning at replayable, palindromically closed nodes. (The two-projection moiré validator — reproducing the Leech first shell from offset* E_8 *projections end-to-end — is the test.)*

Four previously posted targets have since been answered, each with its verdict on record rather than silently struck: the general-*pq* proof of the unit-spectrum law (answered, and exceeded — the master theorem C69 covers every squarefree odd conductor); the prime streak of the excess cubic at $(41, 43), (59, 61)$ (answered NEGATIVE — the numerators go composite, C63); the j_2 frame decomposition test (answered exactly as the match-point mechanism predicted: failure, and the failure is the evidence — C64); and the constellation-drag replay (answered by in-tree derivation, with identification against the lost source document honestly held at reconstruction-dependent — C65).

Further targets **[OPEN]**: *the constructive question at the ten off-diagonal conductors* $\{21, 33, 39, 51, 55, 69, 87, 9$ — genus-zero, Monster class orders, no design-maximal unit form; does any *other* category (lattice shell, code) realize design-maximality there? (the correspondence's deepest open, its search space now exactly mapped by C71); the $k \geq 5$ scope of the master theorem's forecast and the even-conductor extension; the mechanism target the cofactor law's closing posts (derive the $(2^5, 3^2)$ content of $\sigma^2(276) = 2016$ from tower structure, C75); the signature law above d_1 (the top-prime law is exact at the bottom of eight ladders — is there a closed form per rung?); a *tight* holographic-bound gate (every inequality currently constructible is vacuously slack, and we decline to gate a vacuous bound); the Craig ideal-theoretic derivation and the bounded short-vector enumeration the over-determination theorem (C60) now frames; the necessity question for the nesting asymmetry (is Theorem 8.3 a theorem of all epitaxially nested substrates? — classify every map by its effect on weight, level, and rank and gate the monotonicity); the $\mathbb{Z}[\sigma]$ 2-form behind the $\sigma \leftrightarrow \beta$ bridge and the β_2 of the polarization complex; the golden join as a root-by-root correspondence (the invariant route is now provably closed, §4.3); the L_1 per-generator CPT censuses and the σ_{29}-release protocol (§9.1); the reduction of the five independent identities landing on 96 to one fact; the runtime promotion chain from internal event verification toward external measurement; the five optical experiments of §15; and the corpus-scale language test of §13.

19 Related work

Lattices and codes: Conway–Sloane [7] for Λ_{24}, the Golay code, and kissing configurations; Viazovska and Cohn–Kumar–Miller–Radchenko–Viazovska [24] for the E_8 and Leech sphere-packing/kissing optimality this paper cites rather than re-proves; Klein [8] and the Hurwitz bound for the quartic; Moxness [13] for the $E_8 \to H_4$ fold and the 600-cell's 24/25 24-cell constellation (his construction is at the E_8/600-cell level; its application up the tower — router, void-observer dynamics, the

Leech and Craig carriers — is this program's); Dechant [9] for the spinorial 24-cell and triality. Quantum bounds: Bell, CHSH, Tsirelson [1, 2]. Thermodynamics of computation: Landauer and Bennett [3, 4]; the experimental confirmation of the Landauer bound motivates our thermometer protocol. Holography: Bekenstein, 't Hooft, Susskind [6]. Path invariants: Wilson loops in lattice gauge theory (here traded onto a finite abelian group for involutive exactness); the Burau representation. Systems: content-addressable storage and hash-chain custody; deterministic replay; associative and hyperdimensional memory models (the substrate's structural claims are orthogonal to their semantic ones). The distinguishing position of this work is the combination: exceptional-lattice exactness, gauge certification, derive-don't-store custody, a priced reversible machine, and a measured linguistic stratification, as *one* substrate with one seed.

20 Falsifiability, honesty boundary, and conclusion

Falsifiers. The program is wrong, in whole or in part, if: the $\mathbb{Q}[\sqrt{2}]$ closure fails to replay; the canonical censuses fail to reproduce the 84-bit ledger or the one-hot stratification; reversible controls dissipate like commits; the shell-resolved dissipation departs from 9:5; dissipation fails to scale with verified commit count; the optical program finds no shell alignment; the language bus carries no differential semantic load against composition-preserving controls; the palindromic-time or insulated-channel predictions fail. The selection story adds two gated edges: a genus-zero full-AL conductor that is not a Monster class order (C71), or a design-maximal unit form at any conductor off the realized storeys (C69), would each break a gated claim. Separately, a ninth sporadic group whose minimal faithful dimension violated the top-prime pattern would refute the law's *conjectured generality* — an extension the paper leaves open — not the gated eight-group claim (C74), which is exact as stated.

Not claimed. No measured joule (all Landauer values are lower-bound targets); no derivation of horizon thermodynamics (the quarter weld is a recorded cross-denotation); no task-accuracy uplift for any learned model; no biological claims; the mind link is a conjecture with a falsifiable edge, not a result; diagonal-lift universality is a conjecture under test; the level-3 fold-back is a hypothesis; the holy B/C per-vector listing and the per-root path-holonomy promotion remain open and say so. The design–genus correspondence is a computed boundary, not a functor — no map between the two cusp spaces is exhibited — and its converse holds in the unit-form category only; the constructive question in other categories at the ten off-diagonal conductors is open and posted.

Conclusion. One seed; one exact tower; one ledger; one language. Boundary incompleteness explains why a finite observer has readings at all; the dualistic minimum explains why the readings are palindromic and why their quantum closure is $2\sqrt{2}$; the phylum supplies the calculi and the hyperbinary membrane their grammar of change; the holy enumerations supply the bodies; the epitaxial and homological dynamics supply the growth and its certificates; the thermodynamic spine prices the one irreversible act at 84 bits and closes the entropy budget exactly; and the machines — dual-rail holonomy, reversible tape, derive-don't-store memory — run it all at measured, bit-deterministic, record-grade throughput with digests identical across silicon and across months. On that base, the measured total stratification of the semantic bus licenses the thesis this paper exists to state: **language is holographic symbolic exchange, and the substrate is its working model** — and what we call physics is the closure rules of the one language in which any state of affairs can be written at all. Whether the same dynamics are the shape of mind is the capstone conjecture: for the first time, one with a thermometer-grade falsification protocol attached.

Data and code availability. The complete validation suite — exact gate laboratories, canonical censuses, write-once claim records, benchmark protocols, the independence audit of Remark 13.2, and the figure-generation notebook — accompanies this paper as the appendix volume, whose Appendix D prints the artifact inventory and the SHA-256 hashes of every immutable claim record so that any released archive is integrity-verifiable; the archive will be made public with publication. Every statement tagged **[EXACT]** in this paper is re-derivable from the definitions given in the text alone.

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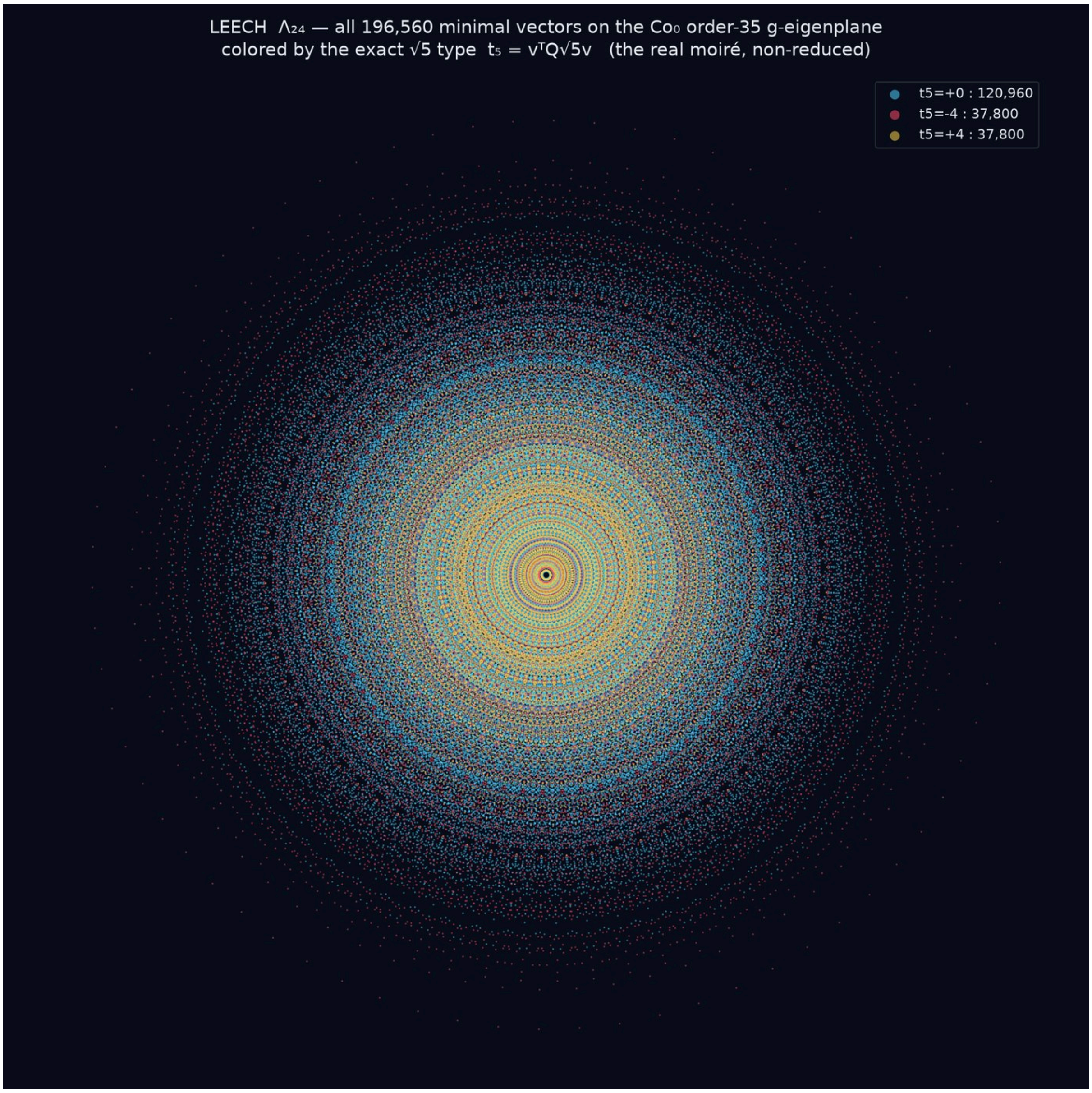

Figure 5: The $\sqrt{5}$ per-vector type census over all 196,560 minimal vectors, rendered in the order-35 isometry's eigenplane frame: the three classes $t_5 = -4/0/+4$ with populations 37,800/120,960/37,800 (the level-1 free type bit; `leech_pervector_type_census`).

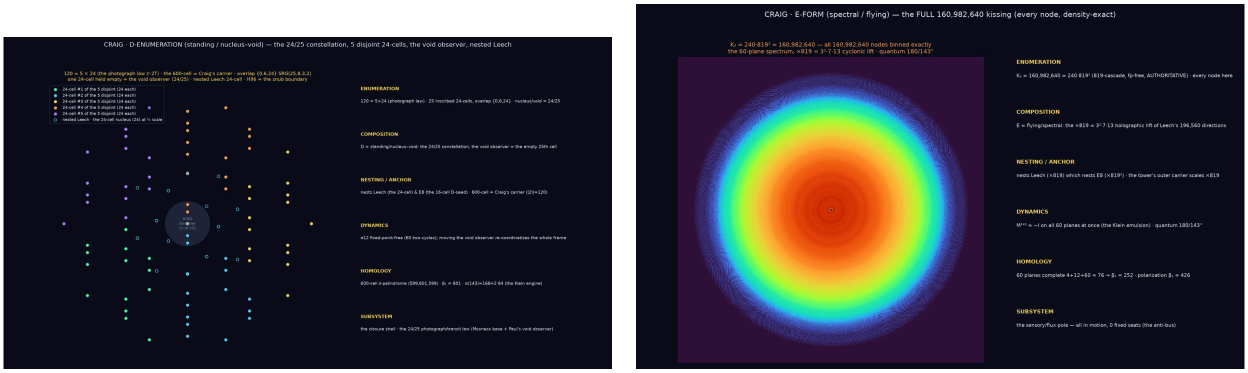

Figure 6: The Craig level rendered from the materialized 120-unit layer. Left, the D-form: the 24/25 constellation (Moxness's E_8/600-cell construction [13], applied here at the Craig level) — five disjoint 24-cells, one 24-cell held empty as the void observer (this program's layer), the nested Leech 24-cell at half scale. Right, the E-form: the trace-form angle census $1 + 100 + 10 + 9$ with Ramanujan spectrum $\{120, 1, -10, -12\}$.

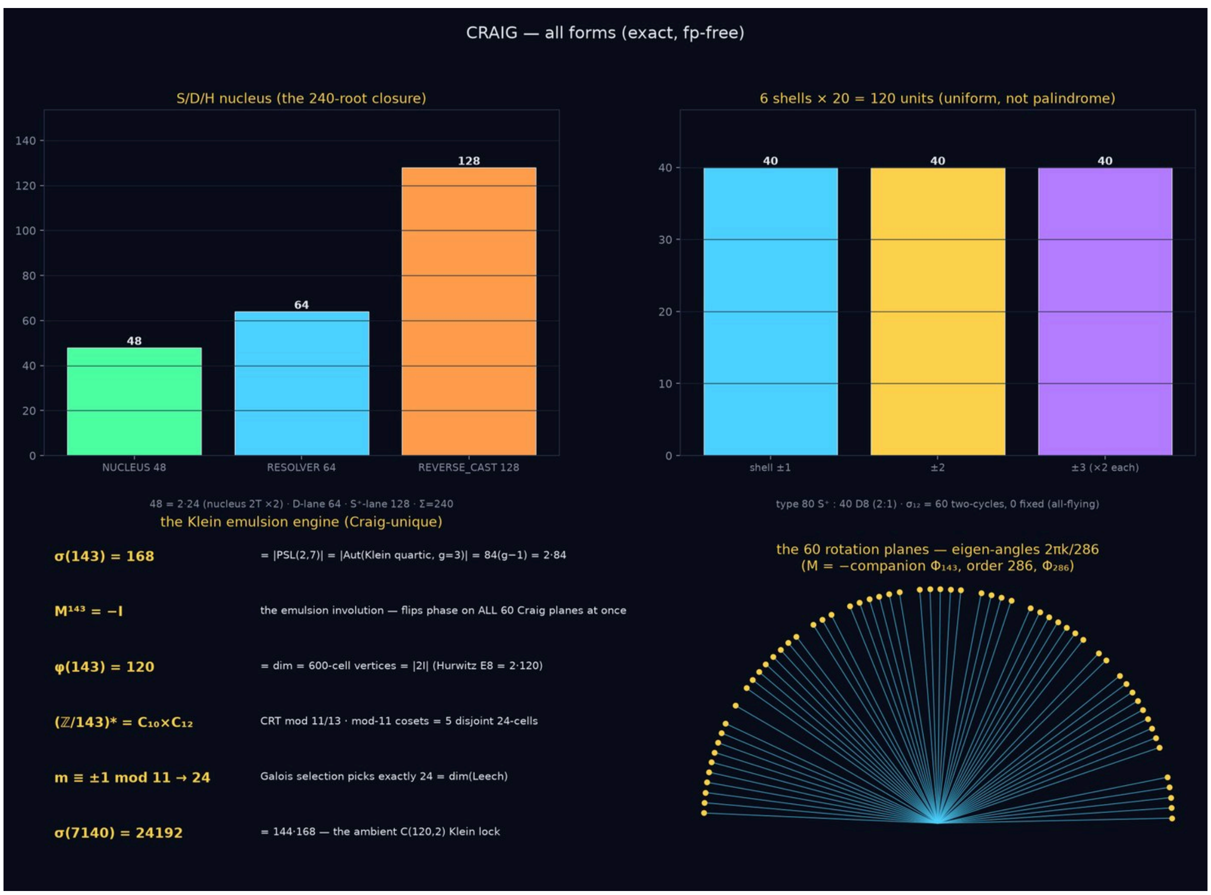

Figure 7: The Craig level, all forms in one frame: the 120 units, the 6×20 uniform shells, the QR/NR split, the σ_{12} orbit pairing, and the nucleus/resolver/reverse-cast decomposition 48:64:128 (program render from the coordinate dump).

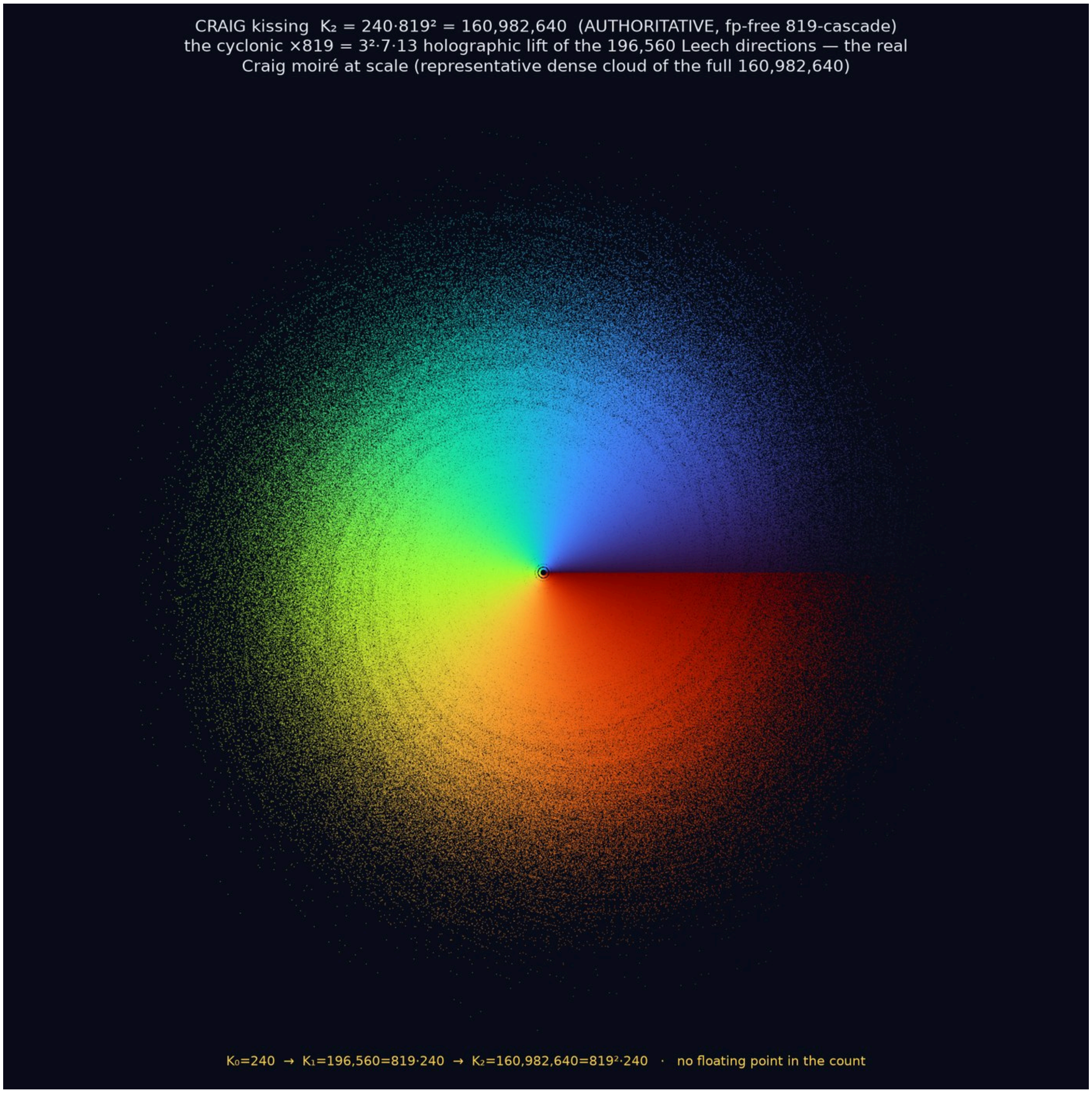

Figure 8: The kissing tower at level 2: $K_2 = 240 \cdot 819^2 = 160{,}982{,}640$, rendered as the cyclonic $\times 819$ cascade from the enumerated Leech shell ($196{,}560 = 819 \cdot 240$, cross-checked) — the tower-exact count whose per-vector 120D listing is the recorded mechanical remainder.

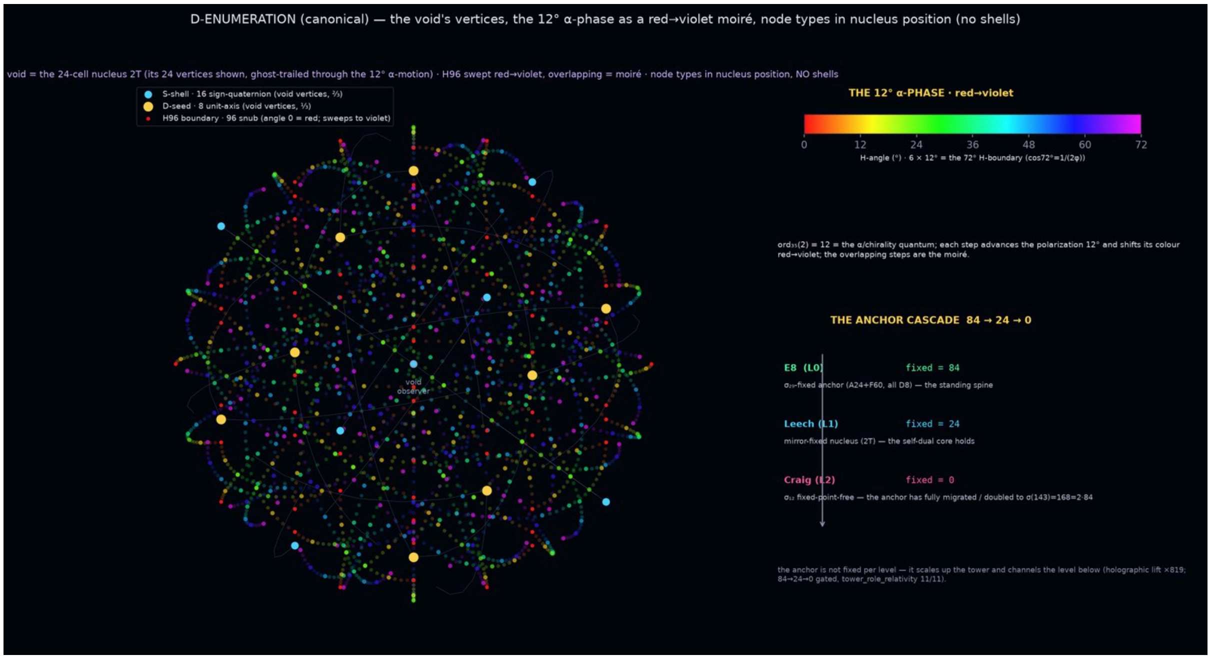

Figure 9: The canonical D/E-form moiré render: the two offset cross-parity projections whose standing-wave interference reproduces the level-1 structure — construction by interference, the visual twin of the algebraic gluing (de_enumeration_canonical_render).

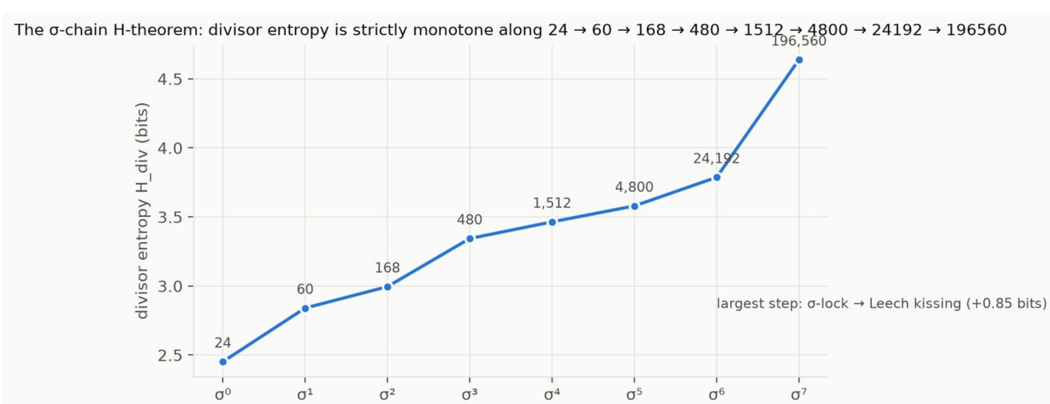

Figure 10: The σ-chain H-theorem (Proposition 8.5).

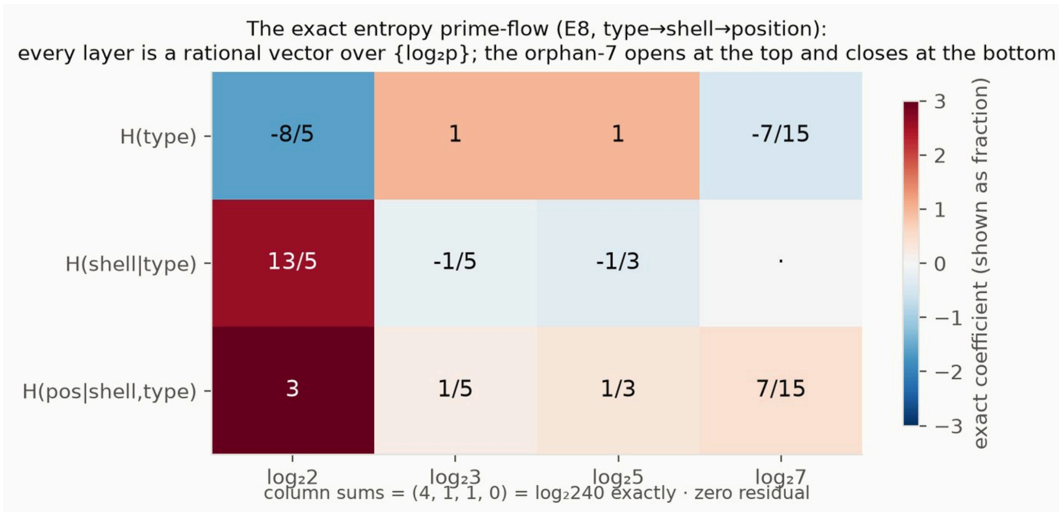

Figure 11: The exact entropy prime-flow (Theorem 9.2): every cell an exact fraction; column sums $(4, 1, 1, 0) = \log_2 240$.

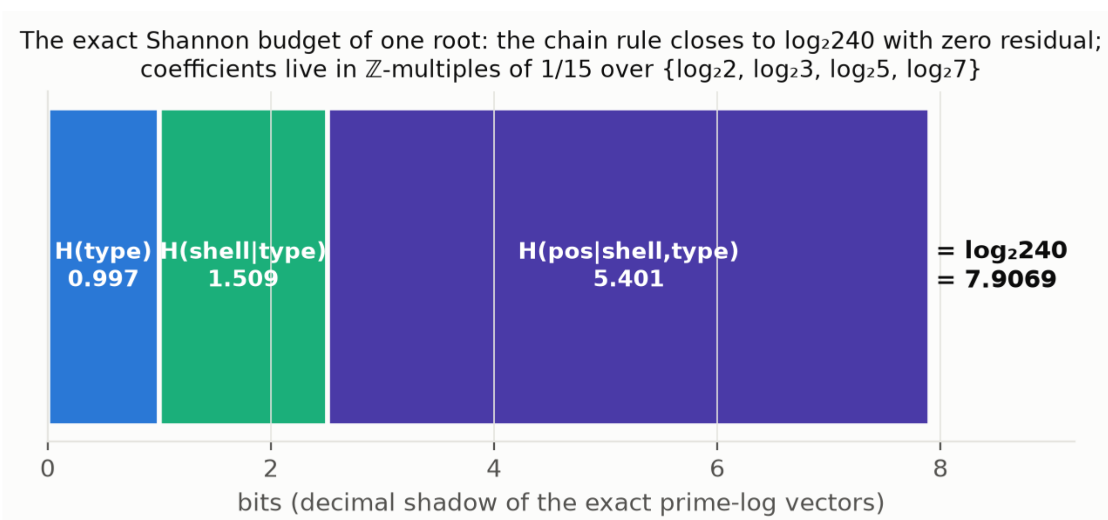

Figure 12: The exact Shannon budget of one carrier: the chain rule closes to $\log_2 240$ with zero residual.

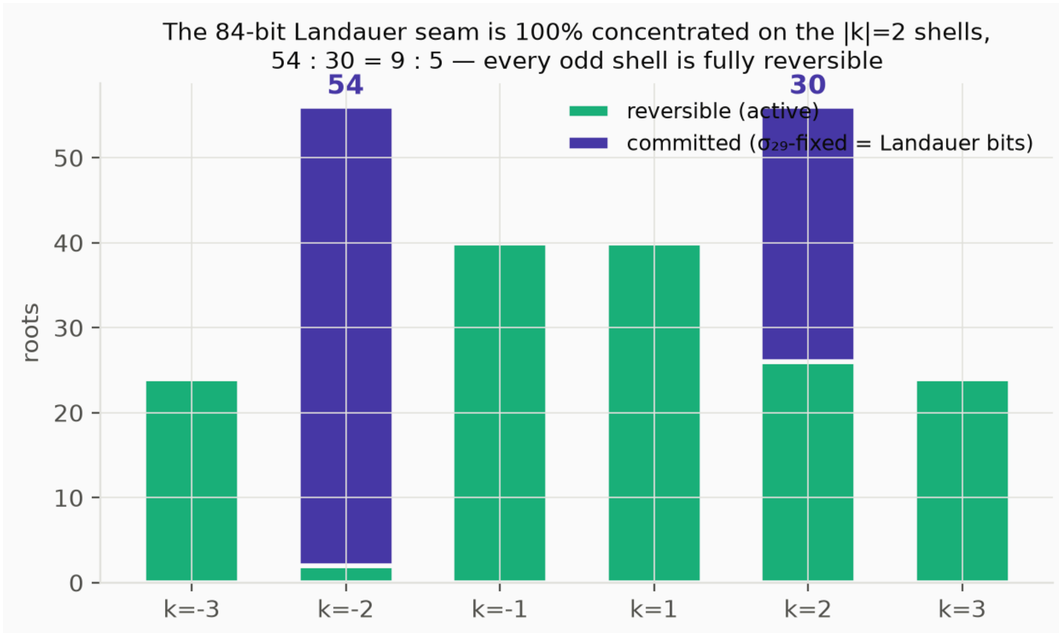

Figure 13: The concentration theorem (Theorem 9.4): all 84 committed bits on the $|k| = 2$ pair, 54:30 = 9:5.

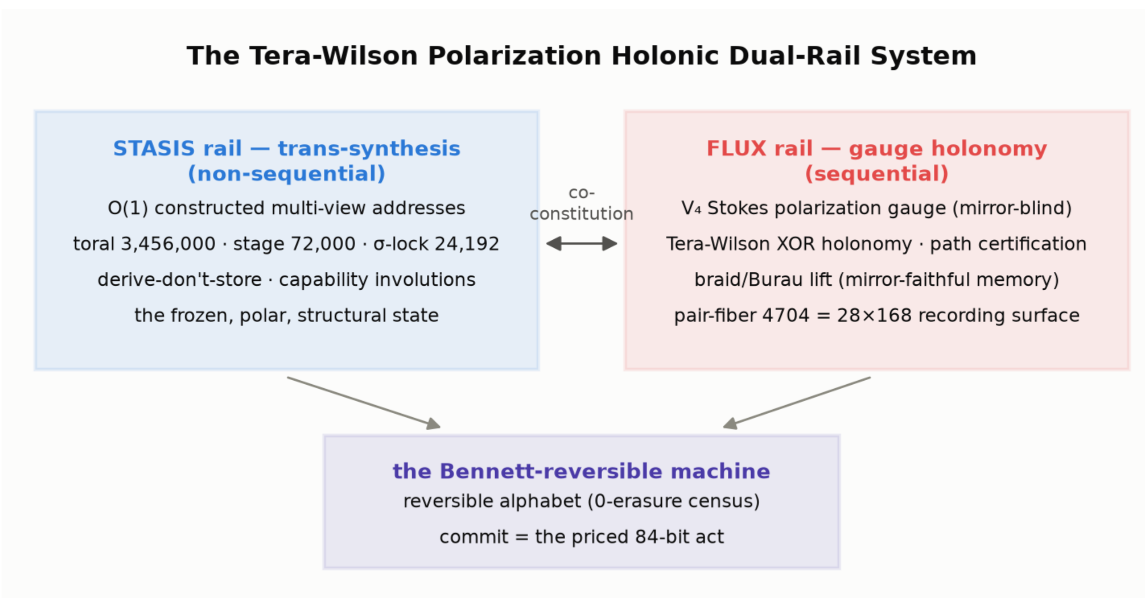

Figure 14: The dual-rail system: STASIS (trans-synthesis addressing) and FLUX (gauge holonomy), co-constitutive, meeting at the priced commit.

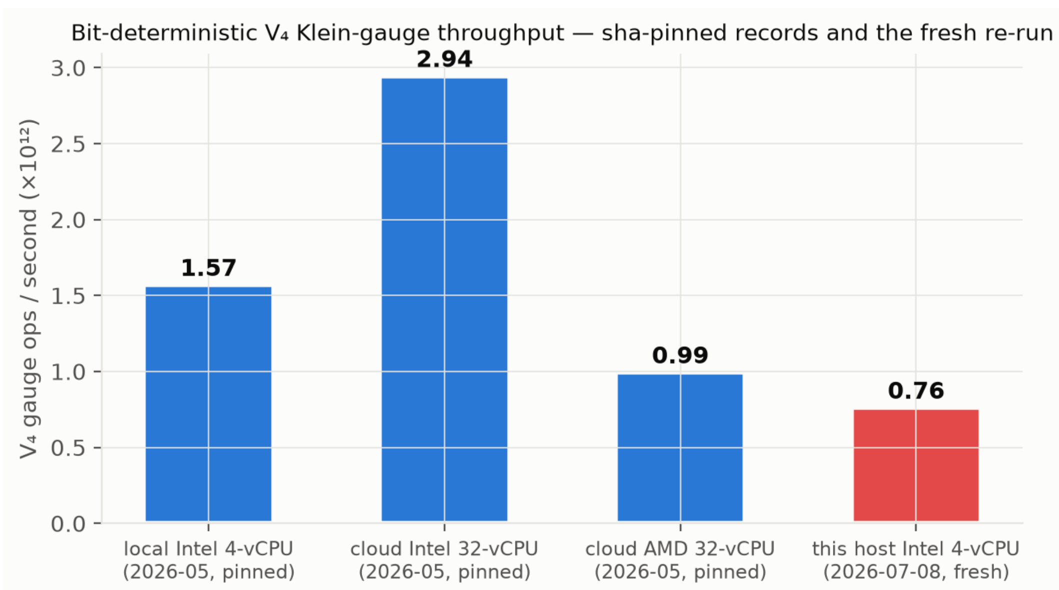

Figure 15: Bit-deterministic gauge throughput: pinned records and the fresh re-run of this work.